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HITTING PROBABILITIES AND HITTING TIMES FOR STOCHASTIC FLUID FLOWS

THE BOUNDED MODEL

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We consider a Markovian stochastic fluid flow model in which the fluid level has a lower bound zero and a positive upper bound. The behavior of the process at the boundaries is modeled by parameters that are different than those in the interior and allow for modeling a range of desired behaviors at the boundaries. We illustrate this with examples. We establish formulas for several time-dependent performance measures of significance to a number of applied probability models. These results are achieved with techniques applied within the fluid flow model directly. This leads to useful physical interpretations, which are presented.

1. INTRODUCTION

We consider a Markovian stochastic fluid flow model, in which the rate of change of a fluid level depends on the state of an underlying Markov chain. As far as the authors

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are aware, such a model was first proposed by Anick, Mitra and Sondhi [6] as a means of studying the occupancy process of a buffer in a data network fed by a number of sources that, at any given time, could be turned on or off. The essence of the model was to approximate the packet arrival process as the input of a fluid modulated by the number of transmitting sources. Similarly, the processing of packets was modeled as a draining of the fluid, at a constant rate.

Since [6], the use of stochastic fluid models has become widespread. They have become highly successful in modeling a number of different aspects of the behavior of telecommunication networks and computer systems [13,23,24]. It has also quickly become evident that these models have tremendous application potential in many other areas, including risk processes in insurance [2,18], manufacturing systems [16], hydro-power generation [9], as well as environmental problems, such as modeling of coral reef resilience [15].

To study this model, Ramaswami [17], da Silva Soares and Latouche [21,22], Ahn and Ramaswami [4], and Ahn, Jeon, and Ramaswami [3] used matrix-analytic methods, which we also apply here. Alternative methods were used by Anick et al. [6], Rogers [20] and Asmussen [7]. These studies produced results suitable for calculating stationary distributions, which has been one area of interest among researchers. Our recent goal has been to develop methods applicable for the calculation of various performance measures of the time taken to traverse sample paths, which is another area of interest. In [10], we derived results for several such measures of the fluid flow model without an upper bound. In [11,12] we described and analyzed several algorithms that can be used to calculate measures for the model in [10]. As we will see, these results are also useful for the calculation of the measures of the model introduced here.

In practice, it will usually be the case that a buffer has an upper boundary. Moreover, the unbounded model might not provide a good approximation for such systems. This has motivated us to consider a bounded model. In this article we focus on *time-related* performance measures, which are of crucial importance to any process involving a control. Ahn et al. [3] and da Silva Soares and Latouche [22] have considered a related bounded model and established results for the stationary distribution, whereas Ahn, Badescu, and Ramaswami [2] reported results for certain time-dependent performance measures. Here, we consider a bounded model with general behavior at the barriers and report results for several time-dependent performance measures not considered in [2].

Before introducing our model, we first define the unbounded process. We use the notation $\mathbb{R}^+$ and $\mathbb{R}$ to denote the set of nonnegative real numbers and the set of real numbers, respectively. The traditional unbounded Markovian fluid flow model is a two-dimensional continuous-time Markov process $\{(M(t), \varphi(t)) : t \in \mathbb{R}^+\}$, where we have the following.

- The level is denoted by $M(t) \in \mathbb{R}$.
- The phase is denoted by $\varphi(t) \in \mathcal{S}$, where $\mathcal{S}$ is some finite set.

- $\mathcal{S}$ is partitioned in the manner $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2$, where the sets $\mathcal{S}_0$, $\mathcal{S}_1$, and $\mathcal{S}_2$ are such that the net rate of input to the infinite fluid buffer, denoted by c_i when $\varphi(t)$ is in state i, is $c_i = 0$ if $i \in \mathcal{S}_0$, $c_i > 0$ if $i \in \mathcal{S}_1$, and $c_i < 0$ if $i \in \mathcal{S}_2$.
- The phase process $\{\varphi(t) : t \in \mathbb{R}^+\}$ is an irreducible Markov chain with infinitesimal generator $\mathbf{T}$.

Let $\mathbf{C}_1$ be the diagonal matrix with $[\mathbf{C}_1]_{ii} = c_i$ for all $i \in \mathcal{S}_1$ and let $\mathbf{C}_2$ be the diagonal matrix with $[\mathbf{C}_2]_{ii} = -c_i$ for all $i \in \mathcal{S}_2$.

We partition the generator $\mathbf{T}$ according to $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2$ so that

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}_{00} & \mathbf{T}_{01} & \mathbf{T}_{02} \\ \mathbf{T}_{10} & \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{T}_{20} & \mathbf{T}_{21} & \mathbf{T}_{22} \end{bmatrix}.$$

The process $(M(t), \varphi(t))$ has the convenient property of being fluid-level homogeneous.

This is not the case for the bounded process $(\widetilde{M}(t), \varphi(t))$, which is defined for $\widetilde{M}(t) \in [0, b]$ and $\varphi(t) \in \mathcal{S}$. To construct this process, we first assume

- that on levels within the interval $(0, b)$, the transition structure of the bounded process $(\widetilde{M}(t), \varphi(t))$ is identical to that of the process $(M(t), \varphi(t))$.

As in [2,3,22], one can construct a bounded model by assuming that $c_i = 0$ for all $i \in \mathcal{S}_2$ when the fluid level is zero and that $c_i = 0$ for all $i \in \mathcal{S}_1$ when the fluid level is b. However, we want to be able to model a wider range of behavior. We achieve this with the introduction of matrixes $\hat{\mathbf{P}}$, $\check{\mathbf{P}}$, $\hat{\mathbf{T}}$, and $\check{\mathbf{T}}$ and sets $\hat{\mathcal{S}}_0$ and $\check{\mathcal{S}}_0$, so that the process $(\widetilde{M}(t), \varphi(t))$ satisfies the following assumptions:

- Once the fluid level reaches b, and does so in some phase $i \in \mathcal{S}_1$, the phase process immediately leaves phase i and moves to some phase $j \in \hat{\mathcal{S}}_0 \cup \mathcal{S}_2$ with probability given by the (i, j)th entry of the stochastic matrix $\hat{\mathbf{P}}$. The net input rate is assumed to be $c_i = 0$ for all $i \in \hat{\mathcal{S}}_0$. From phases in $\hat{\mathcal{S}}_0$, only transitions to phases in $\hat{\mathcal{S}}_0 \cup \mathcal{S}_2$ are allowed. The rate of such transitions is governed by the conservative generator $\hat{\mathbf{T}}$. Once the phase process enters set $\mathcal{S}_2$, the bounded process $(\widetilde{M}(t), \varphi(t))$ behaves in an manner equivalent to $(M(t), \varphi(t))$ again.
- Similarly, once the fluid level reaches zero, and does so in some phase $k \in \mathcal{S}_2$, the phase process immediately leaves phase k and moves to some phase $\ell \in \check{\mathcal{S}}_0 \cup \mathcal{S}_1$ with probability given by the (k, ℓ)th entry of the stochastic matrix $\check{\mathbf{P}}$. The net input rate is assumed to be $c_i = 0$ for all $i \in \check{\mathcal{S}}_0$. From phases in $\check{\mathcal{S}}_0$, only transitions to phases in $\check{\mathcal{S}}_0 \cup \mathcal{S}_1$ are allowed. The rate of such transitions is governed by the conservative generator $\check{\mathbf{T}}$. Once the phase process enters the set $\mathcal{S}_1$, the process $(\widetilde{M}(t), \varphi(t))$ behaves in a manner equivalent to $(M(t), \varphi(t))$.

We partition the matrix $\hat{\mathbf{P}}$ representing the transition $\mathcal{S}_1 \to \hat{\mathcal{S}}_0 \cup \mathcal{S}_2$ and the matrix $\check{\mathbf{P}}$ representing the transition $\mathcal{S}_2 \to \check{\mathcal{S}}_0 \cup \mathcal{S}_1$ so that

$$\hat{\mathbf{P}} = \left[\hat{\mathbf{P}}_{10} \quad \hat{\mathbf{P}}_{12}\right], \qquad \check{\mathbf{P}} = \left[\check{\mathbf{P}}_{20} \quad \check{\mathbf{P}}_{21}\right].$$

Similarly, we let

$$\hat{\mathbf{T}} = \left[\hat{\mathbf{T}}_{00} \quad \hat{\mathbf{T}}_{02}\right], \qquad \check{\mathbf{T}} = \left[\check{\mathbf{T}}_{00} \quad \check{\mathbf{T}}_{01}\right].$$

In the bounded fluid model in [2,3,22], it is assumed that $c_i = 0$ whenever $i \in \mathcal{S}_1$ and $M(t) = b$ or $i \in \mathcal{S}_2$ and $M(t) = 0$. This behavior can be incorporated into our model at the upper boundary by defining $\hat{\mathcal{S}}_0 = \mathcal{S}_0 \cup \mathcal{S}_1$ and letting $\hat{\mathbf{P}}_{10}$, partitioned according to $\mathcal{S}_0 \cup \mathcal{S}_1$, be

$$\hat{\mathbf{P}}_{10} = [\mathbf{0} \quad \mathbf{I}],$$

with an analogous modification applied to the lower boundary. This behavior is a special case of the absorbing barrier described below.

Our model allows us to achieve a range of desired behaviors. Some of these behaviors include the following:

- **Absorbing barrier** This occurs when a process remains at a barrier for a period of time. To achieve this at the upper boundary, we force an immediate transition from $\mathcal{S}_1$ to $\hat{\mathcal{S}}_0$ at the boundary. To implement this, we let $\hat{\mathbf{P}}_{10}$ be a strictly stochastic matrix.
- **Immediate reflection barrier** This is where the process must immediately reflect away from the barrier. There is no possibility of remaining at the barrier while the dynamics of the process alter (such as during a period of overflow of a finite buffer). To implement this at the upper boundary, we prevent the process remaining on the barrier by forcing an immediate transition from $\mathcal{S}_1$ to $\mathcal{S}_2$ at the boundary. To apply this at the upper boundary, we let $\hat{\mathcal{S}}_0 = \emptyset$ and can simply make $\hat{\mathbf{P}}_{12}$ a strictly stochastic matrix.

The article is structured as follows. Section 2, which relates to the unbounded model only, contains preliminary results. These results are essential for the calculation of the matrixes introduced in the subsequent section. Specifically, we establish the expressions for the Laplace–Stieltjes transforms of the times taken to drain/fill the (unbounded) buffer to a given level. In Section 3 we define two matrixes, which are essential for the subsequent derivation of the results for the bounded model. These matrixes are used in the derivation of the Laplace–Stieltjes transforms of the time taken to traverse sample paths corresponding to four different types of behavior, as defined in Section 4. The results of Section 3 are relevant to both the bounded and unbounded models.

We treat the bounded model explicitly in Sections 4–6. The proofs of the main results for this model, in the form of the Laplace–Stieltjes transforms, are achieved by appropriate conditioning, in which sample paths are partitioned into parts corresponding to the four types of behavior defined in Section 4. In Section 4 we establish the results for the first-return times to the initial level. Section 5 contains the results for the draining/filling times. The formulas for the sojourn times in specified sets are given in Section 6. We illustrate the results of this article with numerical examples in Section 7, in which we calculate the probability densities by inverting the Laplace–Stieltjes transforms using the method of Abate and Whitt [1]. This is followed by concluding remarks in Section 8.

2. THE UNBOUNDED MODEL

This section contains preliminary results. Here, and throughout, we assume that $\mathrm{Re}(s) \geq 0$. We consider the unbounded process $(M(t), \varphi(t))$ and introduce matrixes $\hat{\mathbf{G}}^x(s)$ and $\hat{\mathbf{H}}^x(s)$, which record the Laplace–Stieltjes transforms of the time taken to drain the buffer from the initial level $z + x$ to level z and to fill the buffer from z to $z + x$, respectively. Without loss of generality, due to the upward and downward homogeneity of the process, we may assume that $z = 0$. We establish expressions for these two matrixes in Theorem 1. These matrixes are applied in the derivation of the expressions for the two matrixes in Section 3 as presented in Theorem 2.

Let $\theta(z) = \inf\{t > 0 : M(t) = z\}$ be the first passage time to level z in the process $(M(t), \varphi(t))$. For $i, j \in \mathcal{S}_1 \cup \mathcal{S}_2$, $x > 0$, let $\hat{\mathbf{G}}^x(s)$ be the matrix such that

$$[\hat{\mathbf{G}}^x(s)]_{ij} = E[e^{-s\theta(0)}; \theta(0) < \infty \,, \varphi(\theta(0)) = j \mid M(0) = x, \varphi(0) = i]. \qquad \mathbf{(1)}$$

$[\hat{\mathbf{G}}^x(s)]_{ij}$ is the Laplace–Stieltjes transform of the time taken, starting from level x in phase i, for the process $(M(t), \varphi(t))$ to first reach level 0 and do so in phase j.

Similarly, for $i, j \in \mathcal{S}_1 \cup \mathcal{S}_2$, $x > 0$, let $\hat{\mathbf{H}}^x(s)$ be the matrix such that

$$[\hat{\mathbf{H}}^x(s)]_{ij} = E[e^{-s\theta(x)}; \theta(x) < \infty \,, \varphi(\theta(x)) = j \mid M(0) = 0, \varphi(0) = i]. \qquad \mathbf{(2)}$$

$[\hat{\mathbf{H}}^x(s)]_{ij}$ is the Laplace–Stieltjes transform of the time taken, starting from level 0 in phase i, for the process $(M(t), \varphi(t))$ to first reach level x and do so in phase j.

Note that sample paths contributing to $\hat{\mathbf{G}}^x(s)$ can end only in some phase in $\mathcal{S}_2$, whereas sample paths contributing to $\hat{\mathbf{H}}^x(s)$ can end only in some phase in $\mathcal{S}_1$. Consequently, the matrixes $\hat{\mathbf{G}}^x(s)$ and $\hat{\mathbf{H}}^x(s)$ can be partitioned according to $\mathcal{S}_1 \cup \mathcal{S}_2$ so that

$$\hat{\mathbf{G}}^x(s) = \begin{bmatrix} \mathbf{0} & \hat{\mathbf{G}}^x_{12}(s) \\ \mathbf{0} & \hat{\mathbf{G}}^x_{22}(s) \end{bmatrix}, \qquad \hat{\mathbf{H}}^x(s) = \begin{bmatrix} \hat{\mathbf{H}}^x_{11}(s) & \mathbf{0} \\ \hat{\mathbf{H}}^x_{21}(s) & \mathbf{0} \end{bmatrix}.$$

We are interested in expressions for $\hat{\mathbf{G}}^x(s)$ and $\hat{\mathbf{H}}^x(s)$ and establish these in Theorem 1. First, we introduce two matrixes $\hat{\boldsymbol{\Psi}}(s)$ and $\hat{\Xi}(s)$.

Following [10], for $i \in \mathcal{S}_1, j \in \mathcal{S}_2$, let $\hat{\boldsymbol{\Psi}}(s)$ be the matrix such that

$$[\hat{\boldsymbol{\Psi}}(s)]_{ij} = E[e^{-s\theta(z)}; \theta(z) < \infty, \varphi(\theta(z)) = j \mid M(0) = z, \varphi(0) = i]. \qquad \mathbf{(3)}$$

$[\hat{\boldsymbol{\Psi}}(s)]_{ij}$ is the Laplace–Stieltjes transform of the times taken by sample paths that start in phase $i \in \mathcal{S}_1$ at level z and first return to level z in phase $j \in \mathcal{S}_2$.

Similarly, for $i \in \mathcal{S}_2, j \in \mathcal{S}_1$, let $\hat{\Xi}(s)$ be the matrix such that

$$[\hat{\Xi}(s)]_{ij} = E[e^{-s\theta(z)}; \theta(z) < \infty, \varphi(\theta(z)) = j \mid M(0) = z, \varphi(0) = i]. \qquad \textbf{(4)}$$

$[\hat{\Xi}(s)]_{ij}$ is the Laplace–Stieltjes transform of the times taken by sample paths that start in phase $i \in \mathcal{S}_2$ at level z and first return to level z in phase $j \in \mathcal{S}_1$.

We now state the main result of this section. This result follows from the results for an associated model in Ahn and Ramaswami [5, Thm. 2]. We prove this result by an argument within the fluid model itself, by using the direct techniques developed in [10].

THEOREM 1: *For $x \geq 0$, the matrixes $\hat{\mathbf{G}}^x(s)$ and $\hat{\mathbf{H}}^x(s)$, partitioned according to $\mathcal{S}_1 \cup \mathcal{S}_2$, are given by*

$$\hat{\mathbf{G}}^x(s) = \begin{bmatrix} \mathbf{0} & \hat{\mathbf{G}}^x_{12}(s) \\ \mathbf{0} & \hat{\mathbf{G}}^x_{22}(s) \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \hat{\mathbf{\Psi}}(s)e^{\mathbf{B}(s)x} \\ \mathbf{0} & e^{\mathbf{B}(s)x} \end{bmatrix},$$

$$\hat{\mathbf{H}}^x(s) = \begin{bmatrix} \hat{\mathbf{H}}^x_{11}(s) & \mathbf{0} \\ \hat{\mathbf{H}}^x_{21}(s) & \mathbf{0} \end{bmatrix} = \begin{bmatrix} e^{\mathbf{A}(s)x} & \mathbf{0} \\ \hat{\Xi}(s)e^{\mathbf{A}(s)x} & \mathbf{0} \end{bmatrix},$$

where the matrixes $\mathbf{A}(s)$ and $\mathbf{B}(s)$ are given by

$$\mathbf{A}(s) = \mathbf{C}_1^{-1}\left[(\mathbf{T}_{11} - s\mathbf{I}) + \mathbf{T}_{12}\hat{\Xi}(s) - \mathbf{T}_{10}(\mathbf{T}_{00} - s\mathbf{I})^{-1}\{\mathbf{T}_{01} + \mathbf{T}_{02}\hat{\Xi}(s)\}\right],$$

$$\mathbf{B}(s) = \mathbf{C}_2^{-1}\left[(\mathbf{T}_{22} - s\mathbf{I}) + \mathbf{T}_{21}\hat{\mathbf{\Psi}}(s) - \mathbf{T}_{20}(\mathbf{T}_{00} - s\mathbf{I})^{-1}\{\mathbf{T}_{02} + \mathbf{T}_{01}\hat{\mathbf{\Psi}}(s)\}\right].$$

The proof is given in Appendix A.

3. TWO MATRIXES

We begin this section with definitions of $\hat{\mathbf{G}}^{x,y}(s)$ and $\hat{\mathbf{H}}^{x,y}(s)$ and derive the results needed for calculating these matrixes. Then in Section 4 we analyze the bounded model by partitioning sample paths into appropriate sections.

For $i, j \in \mathcal{S}_1 \cup \mathcal{S}_2$ and $0 \leq x < y$, let $\hat{\mathbf{G}}^{x,y}(s)$ be the matrix such that

$$[\hat{\mathbf{G}}^{x,y}(s)]_{ij} = E[e^{-s\theta(0)}; \theta(0) < \theta(y)\,, \varphi(\theta(0)) = j \mid M(0) = x, \varphi(0) = i]. \qquad \textbf{(5)}$$

$[\hat{\mathbf{G}}^{x,y}(s)]_{ij}$ is the Laplace–Stieltjes transform of the time taken, starting from level x in phase i, for the process $(M(t), \varphi(t))$ to first reach level 0 and do so in phase j while avoiding the upper taboo level y. In addition, we define $\hat{\mathbf{G}}^{x,x}(s) = \lim_{y \to x^+} \hat{\mathbf{G}}^{x,y}(s)$.

Similarly, for $i, j \in \mathcal{S}_1 \cup \mathcal{S}_2$ and $0 \leq x < y$, let $\hat{\mathbf{H}}^{x,y}(s)$ be the matrix such that

$$[\hat{\mathbf{H}}^{x,y}(s)]_{ij} = E[e^{-s\theta(y)}; \theta(y) < \theta(0)\,, \varphi(\theta(y)) = j \mid M(0) = x, \varphi(0) = i]. \qquad \textbf{(6)}$$

In addition, we define $\hat{\mathbf{H}}^{x,x}(s) = \lim_{y \to x^+} \hat{\mathbf{H}}^{x,y}(s)$.

Sample paths contributing to $\hat{\mathbf{G}}^{x,y}(s)$ can end only in some phase in $\mathcal{S}_2$, whereas sample paths contributing to $\hat{\mathbf{H}}^{x,y}(s)$ can end only in some phase in $\mathcal{S}_1$. Consequently, the matrixes $\hat{\mathbf{G}}^{x,y}(s)$ and $\hat{\mathbf{H}}^{x,y}(s)$ can be partitioned according to $\mathcal{S}_1 \cup \mathcal{S}_2$ so that

$$\hat{\mathbf{G}}^{x,y}(s) = \begin{bmatrix} \mathbf{0} & \hat{\mathbf{G}}^{x,y}_{12}(s) \\ \mathbf{0} & \hat{\mathbf{G}}^{x,y}_{22}(s) \end{bmatrix}, \qquad \hat{\mathbf{H}}^{x,y}(s) = \begin{bmatrix} \hat{\mathbf{H}}^{x,y}_{11}(s) & \mathbf{0} \\ \hat{\mathbf{H}}^{x,y}_{21}(s) & \mathbf{0} \end{bmatrix}.$$

THEOREM 2: *For $s \neq 0$, $0 \leq x \leq y$,*

$$[\hat{\mathbf{G}}^{x,y}(s) \quad \hat{\mathbf{H}}^{x,y}(s)] = [\hat{\mathbf{G}}^{x}(s) \quad \hat{\mathbf{H}}^{y-x}(s)] \begin{bmatrix} \mathbf{I} & \hat{\mathbf{H}}^{y}(s) \\ \hat{\mathbf{G}}^{y}(s) & \mathbf{I} \end{bmatrix}^{-1}. \tag{7}$$

The above is also true for $s = 0$ when the process $\{(M(t), \varphi(t)) : t \in \mathbb{R}^+\}$ is transient.

PROOF: For all s with $\mathrm{Re}(s) \geq 0$, the following minor modification of a result in [10] (achieved by changing probabilities to the Laplace–Stieltjes transforms), and also derived independently in [19], gives expressions for $\hat{\mathbf{G}}^{x,y}(s)$ and $\hat{\mathbf{H}}^{x,y}(s)$ in terms of the matrixes $\hat{\mathbf{G}}^{y}(s)$, $\hat{\mathbf{H}}^{y}(s)$, $\hat{\mathbf{G}}^{x}(s)$, and $\hat{\mathbf{H}}^{y-x}(s)$, which in turn can be calculated using Theorem 1:

$$[\hat{\mathbf{G}}^{x,y}(s) \quad \hat{\mathbf{H}}^{x,y}(s)] \begin{bmatrix} \mathbf{I} & \hat{\mathbf{H}}^{y}(s) \\ \hat{\mathbf{G}}^{y}(s) & \mathbf{I} \end{bmatrix} = [\hat{\mathbf{G}}^{x}(s) \quad \hat{\mathbf{H}}^{y-x}(s)]. \tag{8}$$

If the process $\{(M(t), \varphi(t)) : t \in \mathbb{R}^+\}$ is null-recurrent, then, by [10], matrixes $\hat{\mathbf{G}}^{y}(0)$ and $\hat{\mathbf{H}}^{y}(0)$ are both stochastic, and so the inverse

$$\begin{bmatrix} \mathbf{I} & \hat{\mathbf{H}}^{y}(0) \\ \hat{\mathbf{G}}^{y}(0) & \mathbf{I} \end{bmatrix}^{-1}$$

does not exist. However, if the process is transient, then, by [10], either $\hat{\mathbf{G}}^{y}(s)$ or $\hat{\mathbf{H}}^{y}(s)$ is substochastic, and so the inverse exists.

Suppose that $s \neq 0$. Then $\hat{\mathbf{G}}^{y}(s)\mathbf{e} < \hat{\mathbf{G}}^{y}(0)\mathbf{e} \leq \mathbf{e}$ and $\hat{\mathbf{H}}^{y}(s)\mathbf{e} < \hat{\mathbf{H}}^{y}(0)\mathbf{e} \leq \mathbf{e}$, where $\mathbf{e}$ is a column of 1s of an appropriate size, and so the inverse exists. Hence, the result follows. ■

For $s = 0$, the matrixes $\hat{\mathcal{G}}^{x,y}(s)$ and $\hat{\mathcal{H}}^{x,y}(s)$ can also be calculated when the process $(M(t), \varphi(t))$ is null-recurrent. This can be achieved as follows. Construct a sequence of transient processes $\{(M(t)^{(\epsilon)}, \varphi(t)^{(\epsilon)})\}$, $\epsilon > 0$, with generators $\mathcal{T}^{(\epsilon)}$ converging uniformly to $\mathcal{T}$, the generator of $(M(t), \varphi(t))$. For each process $\{(M(t)^{(\epsilon)}, \varphi(t)^{(\epsilon)})\}$, let the matrix $[\hat{\mathcal{G}}^{x,y}(s)^{(\epsilon)} \quad \hat{\mathcal{H}}^{x,y}(s)^{(\epsilon)}]$ be defined (and partitioned) in the way analogous to the definition of the matrix $[\hat{\mathcal{G}}^{x,y}(s) \quad \hat{\mathcal{H}}^{x,y}(s)]$ for the process $(M(t), \varphi(t))$. Then apply the result below with $s = 0$.

LEMMA 1: *For any sequence of processes* $\{(M(t)^{(\epsilon)}, \varphi(t)^{(\epsilon)})\}$ *with generators* $\mathcal{T}^{(\epsilon)}$ *converging uniformly to* $\mathcal{T}$, *the generator of* $(M(t), \varphi(t))$, *we have*

$$\lim_{\epsilon\to 0^+} [\hat{\mathcal{G}}^{x,y}(s)^{(\epsilon)} \ \hat{\mathcal{H}}^{x,y}(s)^{(\epsilon)}] = [\hat{\mathcal{G}}^{x,y}(s) \ \hat{\mathcal{H}}^{x,y}(s)]. \quad \textbf{(9)}$$

The proof is given in Appendix B.

4. RETURN JOURNEY TO THE INITIAL LEVEL

We now analyze the bounded model. In this analysis, we use the familiar technique of decomposing sample paths into sections with lower or upper taboos [2,10,19, 22]. Specifically, we partition sample paths into the following four different types of behavior, which are illustrated in Figures 1–4.

1. The fluid level, starting at level c in some phase $i \in \mathcal{S}_1$, first moves to level $d > c$ in some phase $j \in \mathcal{S}_1$ while avoiding the lower taboo level c (Fig. 1).
2. The fluid level, starting at level c in some phase $i \in \mathcal{S}_1$, first returns to level c in some phase $j \in \mathcal{S}_2$ while avoiding the upper taboo level $d > c$ (Fig. 2).
3. The fluid level, starting at level d in some phase $i \in \mathcal{S}_2$, first moves to level $c < d$ in some phase $j \in \mathcal{S}_2$ while avoiding the upper taboo level d (Fig. 3).
4. The fluid level, starting at level d in some phase $i \in \mathcal{S}_2$, first returns to level d in some phase $j \in \mathcal{S}_1$ while avoiding the lower taboo level $c < d$ (Fig. 4).

Observe that the matrixes $\hat{\mathbf{H}}_{11}^{0,d-c}(s)$, $\hat{\mathbf{G}}_{12}^{0,d-c}(s)$, $\hat{\mathbf{G}}_{22}^{d-c,d-c}(s)$, and $\hat{\mathbf{H}}_{21}^{d-c,d-c}(s)$ record the Laplace–Stieltjes transforms of the time taken to traverse sample paths of the types illustrated in Figures 1–4, respectively.

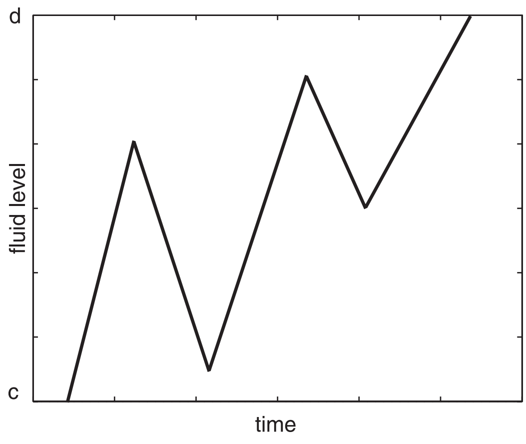

FIGURE 1. Behaviour of type 1.

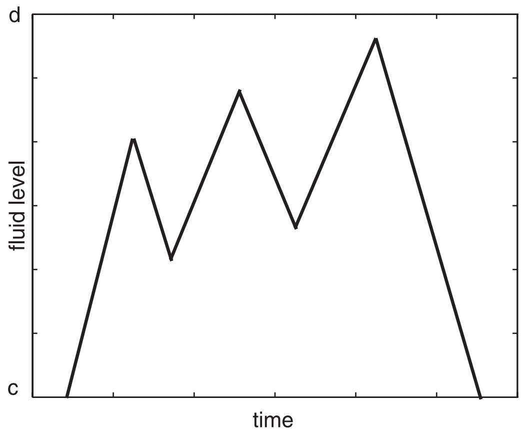

FIGURE 2. Behaviour of type 2.

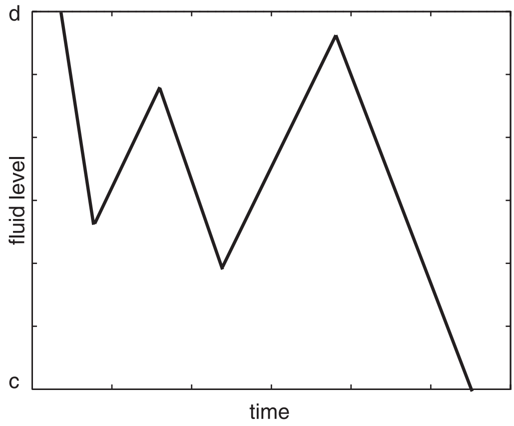

FIGURE 3. Behaviour of type 3.

Because sample paths in the bounded models can be partitioned into parts corresponding to these four types of behavior, we are able to simplify the analysis. In particular, the expressions for the Laplace–Stieltjes transform of the time taken to traverse sample paths in the bounded model can be established now that the Laplace–Stieltjes transforms corresponding to the four types of behavior are known.

Consider the bounded process $(\widetilde{M}(t), \varphi(t))$ and introduce matrixes $\widetilde{\boldsymbol{\Psi}}_z(s)$ and $\widetilde{\Xi}_z(s)$ recording the Laplace–Stieltjes transforms of the time taken to return to the initial level. In Theorem 3 we establish expressions for these matrixes.

Let $\widetilde{\theta}(z) = \inf\{t > 0 : \widetilde{M}(t) = z\}$ be the first passage time to level z in the bounded process $(\widetilde{M}(t), \varphi(t))$. For $z \in [0, b)$, $i \in \mathcal{S}_1$ and $j \in \mathcal{S}_2$, let $\widetilde{\boldsymbol{\Psi}}_z(s)$ be the matrix

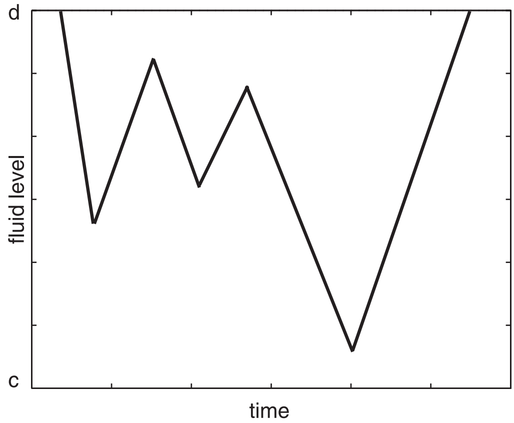

FIGURE 4. Behaviour of type 4.

such that

$$[\widetilde{\mathbf{\Psi}}_z(s)]_{ij} = E[e^{-s\widetilde{\theta}(z)}; \widetilde{\theta}(z) < \infty\ , \varphi(\widetilde{\theta}(z)) = j \mid \widetilde{M}(0) = z, \varphi(0) = i] \tag{10}$$

and let $\widetilde{\mathbf{\Psi}}_b(s) = \lim_{z \to b^-} \widetilde{\mathbf{\Psi}}_z(s)$. Similarly, for $z \in (0, b]$, $i \in \mathcal{S}_2$, and $j \in \mathcal{S}_1$, let $\widetilde{\Xi}_z(s)$ be the matrix such that

$$[\widetilde{\Xi}_z(s)]_{ij} = E[e^{-s\widetilde{\theta}(z)}; \widetilde{\theta}(z) < \infty\ , \varphi(\widetilde{\theta}(z)) = j \mid \widetilde{M}(0) = z, \varphi(0) = i] \tag{11}$$

and let $\widetilde{\Xi}_0(s) = \lim_{z \to 0^+} \widetilde{\Xi}_z(s)$.

Theorem 3 gives expressions for $\widetilde{\mathbf{\Psi}}_z(s)$ and $\widetilde{\Xi}_z(s)$, in terms of matrixes that can be calculated using Theorem 2.

THEOREM 3: *The matrix $\widetilde{\mathbf{\Psi}}_z(s)$ is given by*

$$\widetilde{\mathbf{\Psi}}_z(s) = \hat{\mathbf{G}}_{12}^{0,b-z}(s) + \hat{\mathbf{H}}_{11}^{0,b-z}(s)\hat{\mathbf{W}}(s)\hat{\mathbf{G}}_{22}^{b-z,b-z}(s), \tag{12}$$

where

$$\hat{\mathbf{W}}(s) = [\hat{\mathbf{P}}_{10}(s\mathbf{I} - \hat{\mathbf{T}}_{00})^{-1}\hat{\mathbf{T}}_{02} + \hat{\mathbf{P}}_{12}] \times (\mathbf{I} - \hat{\mathbf{H}}_{21}^{b-z,b-z}(s)[\hat{\mathbf{P}}_{10}(s\mathbf{I} - \hat{\mathbf{T}}_{00})^{-1}\hat{\mathbf{T}}_{02} + \hat{\mathbf{P}}_{12}])^{-1}.$$

The matrix $\widetilde{\Xi}_z(s)$ is given by

$$\widetilde{\Xi}_z(s) = \hat{\mathbf{H}}_{21}^{z,z}(s) + \hat{\mathbf{G}}_{22}^{z,z}(s)\check{\mathbf{W}}(s)\hat{\mathbf{H}}_{11}^{0,z}(s), \tag{13}$$

where

$$\check{\mathbf{W}}(s) = [\check{\mathbf{P}}_{20}(s\mathbf{I} - \check{\mathbf{T}}_{00})^{-1}\check{\mathbf{T}}_{01} + \check{\mathbf{P}}_{21}](\mathbf{I} - \hat{\mathbf{G}}_{12}^{0,z}(s)[\check{\mathbf{P}}_{20}(s\mathbf{I} - \check{\mathbf{T}}_{00})^{-1}\check{\mathbf{T}}_{01} + \check{\mathbf{P}}_{21}])^{-1}.$$

PROOF: We prove the expression for $\widetilde{\boldsymbol{\Psi}}_z(s)$ and the expression for $\widetilde{\Xi}_z(s)$ follow by symmetry. Consider two alternatives for the return journey to the initial level z. The first alternative is that the bounded process $(\widetilde{M}(t), \varphi(t))$, starting from level z in phase $i \in \mathcal{S}_1$, first returns to level z in finite time and does so in phase $j \in \mathcal{S}_2$ while avoiding the upper taboo level b. This type of behavior is illustrated in Figure 2. The Laplace–Stieltjes transform of the time taken for such a journey is the (i,j)th entry of the matrix

$$\hat{\mathbf{G}}_{12}^{0,b-z}(s). \tag{14}$$

The second alternative is that the bounded process $(\widetilde{M}(t), \varphi(t))$, starting from level z in phase $i \in \mathcal{S}_1$, first reaches level b in some phase in $\mathcal{S}_1$ and then returns to level z in finite time and does so in phase $j \in \mathcal{S}_2$. The corresponding Laplace–Stieltjes transform of the time taken for such journey can be obtained by multiplying the matrixes recording transforms, corresponding to the following stages of this journey.

- In the first stage, starting from level z in phase $i \in \mathcal{S}_1$, the process must first reach level b in some phase $k \in \mathcal{S}_1$ while avoiding a lower taboo level z. This type of behavior is illustrated in Figure 1. The Laplace–Stieltjes transform of the time taken to do so is the (i,k)th entry of the matrix

$$\hat{\mathbf{H}}_{11}^{0,b-z}(s). \tag{15}$$

- In the second stage, starting from level b in phase k, either the process enters the set $\mathcal{S}_2$ instantly; that is, a phase change from k to some $\ell \in \mathcal{S}_2$ occurs. The probability of this alternative is given by the (k,ℓ)th entry of the matrix $\hat{\mathbf{P}}_{12}$. Since this transition is instanteneous (elapsed time is zero), the corresponding Laplace–Stieltjes transform is the (k,ℓ)th entry of the matrix $\hat{\mathbf{P}}_{12}$ as well. Alternatively, a phase change from k to some phase in $\hat{\mathcal{S}}_0$ occurs, after which the process spends some finite time in the set $\hat{\mathcal{S}}_0$ and then enters the set $\mathcal{S}_2$ in phase ℓ. By an argument analogous to step 3 in the proof of [10, Lemma 1], the corresponding Laplace–Stieltjes transform of this alternative is the (k,ℓ)th entry of the matrix $\hat{\mathbf{P}}_{10}(s\mathbf{I} - \hat{\mathbf{T}}_{00})^{-1}\hat{\mathbf{T}}_{02}$. Hence, the Laplace–Stieltjes transform of the time taken to first enter the set $\mathcal{S}_2$ and do so in phase ℓ (corresponding to the second stage of the journey) is the (k,ℓ)th entry of the matrix

$$\hat{\mathbf{P}}_{10}(s\mathbf{I} - \hat{\mathbf{T}}_{00})^{-1}\hat{\mathbf{T}}_{02} + \hat{\mathbf{P}}_{12}. \tag{16}$$

  The probability matrix $\hat{\mathbf{P}}_{12}$, which appears in this expression, is the contribution to the Laplace–Stieltjes transform from the paths in which no time elapses.

- In the third stage, the process could return to level b in some phase in $\mathcal{S}_1$ while avoiding the lower taboo level z, which is a type of behavior illustrated in Figure 4. The corresponding Laplace–Stieltjes transform is the appropriate entry of the matrix $\hat{\mathbf{H}}_{21}^{b-z,b-z}(s)$. Then the process could possibly spend some time on level b before entering set $\mathcal{S}_2$. The corresponding Laplace–Stieltjes transform was given in the second stage as the appropriate entry of the matrix $\hat{\mathbf{P}}_{10}(s\mathbf{I}-\hat{\mathbf{T}}_{00})^{-1}\hat{\mathbf{T}}_{02}+\hat{\mathbf{P}}_{12}$. The process can do this any number of times, including none. Hence, the Laplace–Stieltjes transform of the time taken during the third stage of the journey, ending in some phase $\ell' \in \mathcal{S}_2$, is the $(\ell\ell')$th entry of the matrix

$$\sum_{m=0}^{\infty}(\hat{\mathbf{H}}_{21}^{b-z,b-z}(s)[\hat{\mathbf{P}}_{10}(s\mathbf{I}-\hat{\mathbf{T}}_{00})^{-1}\hat{\mathbf{T}}_{02}+\hat{\mathbf{P}}_{12}])^{m}. \tag{17}$$

Because $\mathbf{T}$ is irreducible, the fluid can reach level 0 starting from any phase in level b, which implies that the sum in (17) converges for $\mathrm{Re}(s) \geq 0$, and can be written as

$$(\mathbf{I}-\hat{\mathbf{H}}_{21}^{b-z,b-z}(s)[\hat{\mathbf{P}}_{10}(s\mathbf{I}-\hat{\mathbf{T}}_{00})^{-1}\hat{\mathbf{T}}_{02}+\hat{\mathbf{P}}_{12}])^{-1}. \tag{18}$$

- In the fourth stage, starting from level b in phase ℓ', the process first reaches level z in finite time and does so in phase j while avoiding the upper taboo level b. This type of behavior is illustrated in Figure 3. The Laplace–Stieltjes transform of the time taken during the fourth stage of the journey is the (ℓ',j)th entry of the matrix
$$\hat{\mathbf{G}}_{22}^{b-z,b-z}(s). \tag{19}$$

By (15)–(19), the Laplace–Stieltjes transform of the second alternative is the (i,j)th entry of the following matrix:

$$\begin{aligned}&\hat{\mathbf{H}}_{11}^{0,b-z}(s)\left[\hat{\mathbf{P}}_{10}(s\mathbf{I}-\hat{\mathbf{T}}_{00})^{-1}\hat{\mathbf{T}}_{02}+\hat{\mathbf{P}}_{12}\right]\\ &\quad\times\left(\mathbf{I}-\hat{\mathbf{H}}_{21}^{b-z,b-z}(s)[\hat{\mathbf{P}}_{10}(s\mathbf{I}-\hat{\mathbf{T}}_{00})^{-1}\hat{\mathbf{T}}_{02}+\hat{\mathbf{P}}_{12}]\right)^{-1}\hat{\mathbf{G}}_{22}^{b-z,b-z}(s).\end{aligned} \tag{20}$$

The sum of matrixes (14) and (20) gives the result. ■

COROLLARY 1: *The matrixes* $\widetilde{\boldsymbol{\Psi}}_b(s)$ *and* $\widetilde{\Xi}_0(s)$ *are given by*

$$\begin{aligned}\widetilde{\boldsymbol{\Psi}}_b(s) &= [\hat{\mathbf{P}}_{10}(s\mathbf{I}-\hat{\mathbf{T}}_{00})^{-1}\hat{\mathbf{T}}_{02}+\hat{\mathbf{P}}_{12}],\\ \widetilde{\Xi}_0(s) &= [\check{\mathbf{P}}_{20}(s\mathbf{I}-\check{\mathbf{T}}_{00})^{-1}\check{\mathbf{T}}_{01}+\check{\mathbf{P}}_{21}].\end{aligned}$$

PROOF: It follows from their definitions that $\hat{\mathbf{H}}_{11}^{0,0}(s)=\mathbf{I}$, $\hat{\mathbf{G}}_{22}^{0,0}(s)=\mathbf{I}$, $\hat{\mathbf{H}}_{21}^{0,0}(s)=0$, and $\hat{\mathbf{G}}_{12}^{0,0}(s)=0$. The result follows by inserting these into (12) and (13). ■

5. DRAINING/FILLING TIMES

In this section we consider the bounded process $(\widetilde{M}(t),\varphi(t))$ and introduce matrixes $\widetilde{\mathbf{G}}^{z|x}(s)$ and $\widetilde{\mathbf{H}}^{z|x}(s)$ recording the Laplace–Stieltjes transforms of the time taken to drain the buffer from level $z+x$ to z and fill the buffer from level z to level $z+x$, respectively. Throughout this section we assume that $z \in [0,b]$ and $x \in [0,b-z]$. We establish expressions for these matrixes in Theorem 4. We begin with definitions of $\widetilde{\mathbf{G}^{z|x}}(s)$ and $\widetilde{\mathbf{H}^{z|x}}(s)$.

For $z \in [0,b)$, $0 < x \le b-z$, and $i,j \in \mathcal{S}_1 \cup \mathcal{S}_2$, let $\widetilde{\mathbf{G}^{z|x}}(s)$ be the matrix such that

$$[\widetilde{\mathbf{G}^{z|x}}(s)]_{ij} = E[e^{-s\widetilde{\theta}(z)}; \widetilde{\theta}(z) < \infty \,, \varphi(\widetilde{\theta}(z)) = j \mid M(0) = z+x, \varphi(0) = i]. \quad \textbf{(21)}$$

In addition, we define $\widetilde{\mathbf{G}^{z|0}}(s) = \lim_{x\to 0^+} \widetilde{\mathbf{G}^{z|x}}(s)$ and $\widetilde{\mathbf{G}^{b|0}}(s) = \lim_{z\to b^-} \widetilde{\mathbf{G}^{z|0}}(s)$.

Similarly, for $z \in [0,b)$, $0 < x \le b-z$, and $i,j \in \mathcal{S}_1 \cup \mathcal{S}_2$, let $\widetilde{\mathbf{H}^{z|x}}(s)$ be the matrix such that

$$[\widetilde{\mathbf{H}^{z|x}}(s)]_{ij} = E[e^{-s\widetilde{\theta}(z+x)}; \widetilde{\theta}(z+x) < \infty \,, \varphi(\widetilde{\theta}(z+x)) = j \mid M(0) = z, \varphi(0) = i]. \quad \textbf{(22)}$$

In addition, we define $\widetilde{\mathbf{H}^{z|0}}(s) = \lim_{x\to 0^+} \widetilde{\mathbf{H}^{z|x}}(s)$ and $\widetilde{\mathbf{H}^{b|0}}(s) = \lim_{z\to b^-} \widetilde{\mathbf{H}^{z|0}}(s)$.

We establish expressions for $\widetilde{\mathbf{G}^{z|x}}(s)$ and $\widetilde{\mathbf{H}^{z|x}}(s)$ below. Expressions for the matrixes appearing in them have been established in Theorems 2 and 3.

THEOREM 4: *The matrixes $\widetilde{\mathbf{G}^{z|x}}(s)$ and $\widetilde{\mathbf{H}^{z|x}}(s)$, partitioned according to $\mathcal{S}_1 \cup \mathcal{S}_2$, are given by*

$$\widetilde{\mathbf{G}^{z|x}}(s) = \begin{bmatrix} \mathbf{0} & \widetilde{\mathbf{G}_{12}^{z|x}}(s) \\ \mathbf{0} & \widetilde{\mathbf{G}_{22}^{z|x}}(s) \end{bmatrix}, \qquad \widetilde{\mathbf{H}^{z|x}}(s) = \begin{bmatrix} \widetilde{\mathbf{H}_{11}^{z|x}}(s) & \mathbf{0} \\ \widetilde{\mathbf{H}_{21}^{z|x}}(s) & \mathbf{0} \end{bmatrix},$$

where

$$\widetilde{\mathbf{G}_{12}^{z|x}}(s) = \widetilde{\mathbf{\Psi}_{z+x}}(s)\widetilde{\mathbf{G}_{22}^{z|x}}(s) \quad \textit{for } z+x \in [0,b],$$

$$\widetilde{\mathbf{G}_{22}^{z|x}}(s) = \begin{cases} \left[\mathbf{I} - \hat{\mathbf{H}}_{21}^{x,x}(s)\widetilde{\mathbf{\Psi}_{z+x}}(s)\right]^{-1} \hat{\mathbf{G}}_{22}^{x,x}(s) & \textit{for } z+x \in (0,b], \\ \mathbf{I} & \textit{for } z+x = 0, \end{cases}$$

$$\widetilde{\mathbf{H}_{21}^{z|x}}(s) = \widetilde{\Xi}_z(s)\widetilde{\mathbf{H}_{11}^{z|x}}(s) \quad \textit{for } z \in [0,b],$$

$$\widetilde{\mathbf{H}_{11}^{z|x}}(s) = \begin{cases} \left[\mathbf{I} - \hat{\mathbf{G}}_{12}^{0,x}(s)\,\widetilde{\Xi}_z(s)\right]^{-1} \hat{\mathbf{H}}_{11}^{0,x}(s) & \textit{for } z \in [0,b) \\ \mathbf{I} & \textit{for } z = b. \end{cases}$$

PROOF: We prove the expression for $\widetilde{\mathbf{G}^{z|x}}(s)$, the expression for $\widetilde{\mathbf{H}^{z|x}}(s)$ follows by symmetry. First, suppose that the bounded process $(\widetilde{M}(t), \varphi(t))$ starts from level $z+x \in [z,b]$ in phase $i \in \mathcal{S}_1$. The Laplace–Stieltjes transform of the time taken to hit level z in finite time and do so in phase $j \in \mathcal{S}_2$, is the (i,j)th entry of the matrix $\widetilde{\mathbf{G}_{12}^{z|x}}(s)$ and can be calculated by multiplying transforms corresponding to the following stages of this journey:

- In the first stage, the process first returns to level $z+x$ in finite time and does so in some phase $k \in \mathcal{S}_2$. The Laplace–Stieltjes transform of the time taken to do so is the (i,k)th entry of the matrix $\widetilde{\mathbf{\Psi}_{z+x}}(s)$.
- In the second stage, starting from level $z+x$ in phase $k \in \mathcal{S}_2$, the process first hits level z in finite time and does so in phase j. The Laplace–Stieltjes transform of the time taken to do so is the (k,j)th entry of the matrix $\widetilde{\mathbf{G}_{22}^{z|x}}(s)$.

Consequently, the formula for $\widetilde{\mathbf{G}_{12}^{z|x}}(s)$ follows.

Now, suppose that the bounded process $(\widetilde{M}(t), \varphi(t))$ starts from level $(z+x) \in (z,b)$ in phase $i \in \mathcal{S}_2$. The Laplace–Stieltjes transform of the time taken to first hit level z in finite time and do so in phase $j \in \mathcal{S}_2$ is the (i,j)th entry of the matrix $\widetilde{\mathbf{G}_{22}^{z|x}}(s)$, with the following stages of this journey:

- In the first stage, the process can first return to level $z+x$ in finite time in some phase in $k \in \mathcal{S}_1$ while avoiding a lower taboo level z, which is the type of behavior illustrated in Figure 4. If this occurs then, starting from level $z+x$ in phase $k \in \mathcal{S}_1$, it must first return to level $z+x$ in finite time and in some phase $\ell \in \mathcal{S}_2$. It can do so any number of times, including none. The Laplace–Stieltjes transform of time corresponding to the first stage is the (i,ℓ)th entry of the matrix $\sum_{m=0}^{\infty}\left(\hat{\mathbf{H}}_{21}^{x,x}(s)\widetilde{\mathbf{\Psi}_{z+x}}(s)\right)^m$. For $\mathrm{Re}(s) \geq 0$, the series converges by the same argument that we used in the proof of Theorem 3 and the matrix can be written as $\left(\mathbf{I} - \hat{\mathbf{H}}_{21}^{x,x}(s)\widetilde{\mathbf{\Psi}_{z+x}}(s)\right)^{-1}$.
- In the second stage, starting from level $z+x$ in phase ℓ, the process must first hit level z in phase j while avoiding the upper taboo level $z+x$, which is the type of behavior illustrated in Figure 3. The Laplace–Stieltjes transform of time corresponding to the second stage is the (ℓ,j)th entry of the matrix $\hat{\mathbf{G}}_{22}^{x,x}(s)$.

Consequently, the formula for $\widetilde{\mathbf{G}_{22}^{z|x}}(s)$ follows. The proof for the case $z+x=b$ is very similar, and when $z+x=0$, it follows similarly to that of Corollary 1. We omit the details. ■

6. SOJOURN TIMES IN SPECIFIED SETS

In this section we consider the bounded process $(\widetilde{M}(t), \varphi(t))$ and introduce the matrix $\widetilde{\mathbf{\Psi}_z^y}(s)$ recording the Laplace–Stieltjes transforms of the time that the process spends above some level y during its return journey to the initial level z. We establish the expression for this matrix in Theorem 5. First, we define $\widetilde{\mathbf{\Psi}_z^y}(s)$.

For $0 \leq z < y < b$, let U^y be the random variable capturing the time spent, by the process $(\widetilde{M}(t), \varphi(t))$ starting from level z, above level y before first returning to level z; that is,

$$U^y = \int_{t=0}^{\widetilde{\theta}(z)} I(\widetilde{M}(t) > y)\, dt. \tag{23}$$

For $i \in \mathcal{S}_1$ and $j \in \mathcal{S}_2$, let $\widetilde{\Psi}_z^y(s)$ be the matrix such that

$$[\widetilde{\Psi}_z^y(s)]_{ij} = E[e^{-sU^y}; U^y < \infty, \varphi(\widetilde{\theta}(z)) = j | \widetilde{M}(0) = z, \varphi(0) = i]. \tag{24}$$

$[\widetilde{\Psi}_z^y(s)]_{ij}$ is the Laplace–Stieltjes transform of the time spent by the process $(\widetilde{M}(t), \varphi(t))$ above level y before the process first returns to level z and does so in phase j, starting from level z in phase i. Additionally, we introduce the notation $\widetilde{\Upsilon}_z^y = -\lim_{s\to 0^+} d/ds\, \widetilde{\Psi}_z^y(s)$.

In Theorem 5 we give an expression for $\widetilde{\Psi}_z^y(s)$. These two results are obtained by arguments similar to those in the proofs of results in [10, Sect. 5], with appropriate modifications.

THEOREM 5: *The matrix* $\widetilde{\Psi}_z^y(s)$ *is given by*

$$\widetilde{\Psi}_z^y(s) = \hat{\mathbf{G}}_{12}^{0,y-z}(0) + \hat{\mathbf{H}}_{11}^{0,y-z}(0)\widetilde{\Psi}_y(s)\left(\mathbf{I} - \hat{\mathbf{H}}_{21}^{y-z,y-z}(0)\widetilde{\Psi}_y(s)\right)^{-1} \hat{\mathbf{G}}_{22}^{y-z,y-z}(0).$$

PROOF: Note that $\hat{\mathbf{G}}_{12}^{0,y-z}(0)$ is a constant with respect to the transform variable s. Its inclusion in the formula is justified as follows. Assume that starting from level $z < y$ in phase $i \in \mathcal{S}_1$ at time 0, the bounded process $(\widetilde{M}(t), \varphi(t))$ first returns to level z in phase $j \in \mathcal{S}_2$. Suppose that the process spends no time above level y on this sample path, and so y is the upper taboo level. This type of behavior is illustrated in Figure 2. The probability of this event is the (i, j)th entry of the matrix

$$\hat{\mathbf{G}}_{12}^{0,y-z}(0). \tag{25}$$

Hence, the probability matrix $\hat{\mathbf{G}}_{12}^{0,y-z}(0)$ is the contribution to the total Laplace–Stieltjes transform from the paths in which no time is spent above level y.

Alternatively, assume that the process spends some time above level y. Then, the following three stages must occur:

- In the first stage, the process must first reach level y and do so in some phase $k \in \mathcal{S}_1$ while avoiding a lower taboo level z, which is the type of behavior illustrated in Figure 1. The probability of this is $[\hat{\mathbf{H}}_{11}^{0,y-z}(0)]_{ik}$.
- In the second stage, starting from level y in phase k, the process must return to level y after spending some time above level y and then can hit level y again, and do so any number of times, including none. By using an argument analogous to that in the proof of [10, Thm. 5] and assuming that the second

stage of the process ends in some phase $\ell \in \mathcal{S}_2$, we have that the Laplace–Stieltjes transform of the time spent above level y during the second stage is the (k, ℓ)th entry of the matrix

$$\widetilde{\mathbf{\Psi}}_y(s) \sum_{m=0}^{\infty} \left(\hat{\mathbf{H}}_{21}^{y-z,y-z}(0)\widetilde{\mathbf{\Psi}}_y(s)\right)^m = \widetilde{\mathbf{\Psi}}_y(s)\left(\mathbf{I} - \hat{\mathbf{H}}_{21}^{y-z,y-z}(0)\widetilde{\mathbf{\Psi}}_y(s)\right)^{-1}.$$

- In the third stage, starting from level y in phase ℓ, the process must first hit level z and do so in phase j while avoiding the upper taboo level y. This type of behavior is illustrated in Figure 3. The probability of this is $[\hat{\mathbf{G}}_{22}^{y-z,y-z}(0)]_{\ell j}$.

Consequently, the Laplace–Stieltjes transform of the second alternative is the (i, j)th entry of the matrix

$$\hat{\mathbf{H}}_{11}^{0,y-z}(0)\widetilde{\mathbf{\Psi}}_y(s)\left(\mathbf{I} - \hat{\mathbf{H}}_{21}^{y-z,y-z}(0)\widetilde{\mathbf{\Psi}}_y(s)\right)^{-1}\hat{\mathbf{G}}_{22}^{y-z,y-z}(0). \quad \textbf{(26)}$$

Because $\widetilde{\mathbf{\Psi}}_z^y(s)$ is the sum of (25) and (26), the result follows. ■

7. EXAMPLES

In this section we consider three different examples, in which the unbounded process has upward drift, has downward drift, and is null-recurrent, respectively. We present these examples to explore the various behaviors that this model can handle on the boundaries—in this case, specifically the upper boundary. In so doing, particularly when considering time-based performance measures, we note that the unbounded model—although being easier to analyze—can provide very inaccurate results.

Let $\psi(t)_{ij}$ be the probability density of first return to the initial level being at time t and in phase $j \in \mathcal{S}_2$, assuming that the process starts in phase $i \in \mathcal{S}_1$ in level $z = 0$; that is,

$$[\widetilde{\mathbf{\Psi}}_0(s)]_{ij} = \int_0^{\infty} e^{-st}\psi(t)_{ij}\,dt.$$

As $\widetilde{\mathbf{\Psi}}_z(s)$ can be efficiently calculated using the results given in this article, we can apply the method of Abate and Whitt [1] for the numerical inversion of Laplace transforms in order to obtain $\psi(t)_{ij}$ from $\widetilde{\mathbf{\Psi}}_z(s)$, for $t > 0$.

Below, we compute the value of $\psi(t)_{ij}$ in three simple examples. For simplicity, in all these examples we let $\mathcal{S}_1 = \{1\}$, $\mathcal{S}_2 = \{2\}$, $c_1 = 1$, $c_2 = -1$, and the upper boundary $b = 10$. For comparison, we also compute the analogous density $\psi(t)_{ij}$ for the unbounded process $(M(t), \varphi(t))$, using Algorithm 4 for $\mathbf{\Psi}(s)$ in [12]. A significant difference between the values for the two densities illustrates the unsurprising fact that using the latter model as an approximation of the former model for processes in which the upper boundary exists is not an accurate approach.

In all of these examples, we also examine how changes in the parameters $\hat{\mathbf{P}}$ and $\hat{\mathbf{T}}$ affect the shape of the probability density $\psi(t)_{ij}$. The purpose of this is to illustrate that by having such parameters, one can model a wide range of behaviors.

Example 1: Let

$$\mathbf{T} = \begin{bmatrix} -1 & 1 \\ 1.5 & -1.5 \end{bmatrix}.$$

The unbounded process $(M(t), \varphi(t))$ with generator $\mathbf{T}$ is transient with an upward drift [10]. We construct three different bounded processes $(\widetilde{M}(t), \varphi(t))$. Their behavior is defined by the following parameters:

(a) $\hat{\mathbf{P}} = [1, 0]$, $\hat{\mathbf{T}} = [-10, 10]$ (absorbing)
(b) $\hat{\mathbf{P}} = [1, 0]$, $\hat{\mathbf{T}} = [-3, 3]$ (absorbing)
(c) $\hat{\mathbf{P}} = [0, 1]$ (immediate reflection)

The results are given in Figures 5 and 6. Note, as expected, the very significant difference for the probability densities $\psi(t)_{12}$ between the bounded and the unbounded

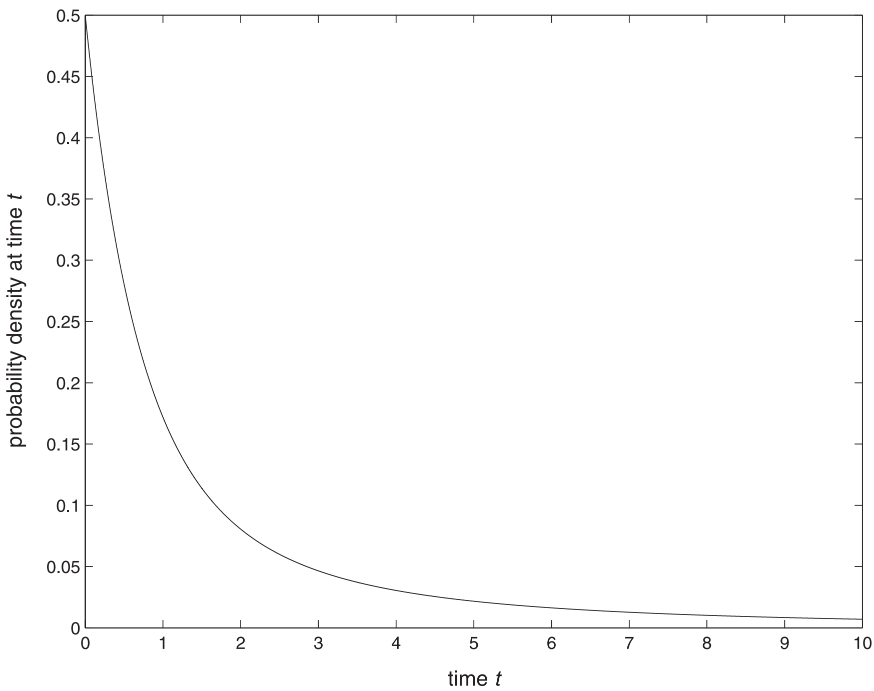

FIGURE 5. Probability density $\psi(t)_{12}$ for the unbounded model in Example 1.

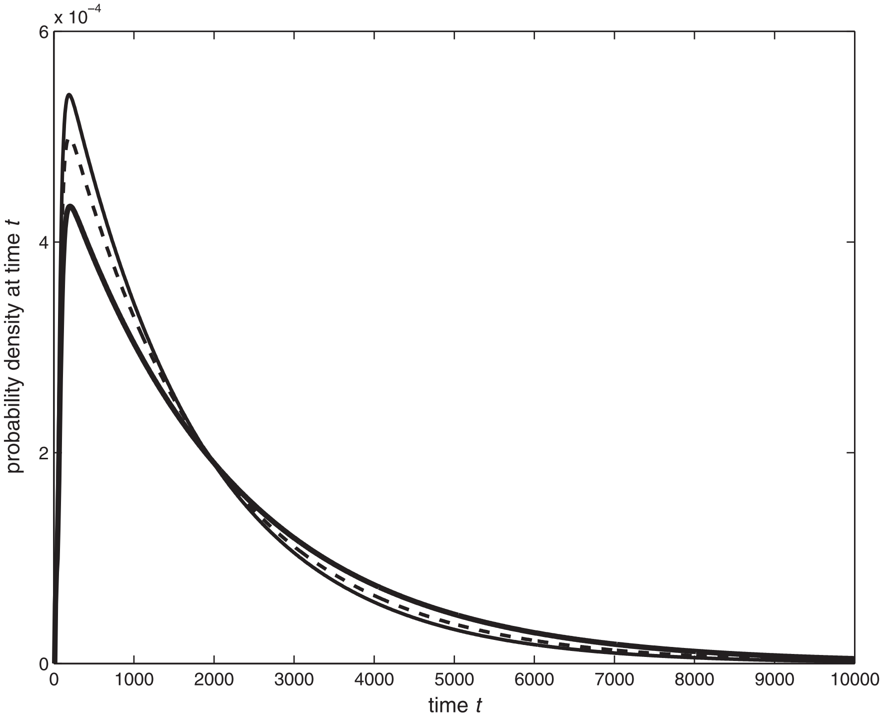

FIGURE 6. Probability densities $\psi(t)_{12}$ for the bounded models in Example 1: (a) dashed line, (b) thick solid line, (c) solid line.

models. The probability, assuming the process starts in phase 1 in level 0, of the first return to level 0 (and doing so in phase 2), for the unbounded and bounded models are $[\boldsymbol{\Psi}]_{12} = 0.6667$ and $[\widetilde{\boldsymbol{\Psi}_0}]_{12} = 1$, respectively.

The probability densities for the bounded models (a)–(c) vary as well. In model (c), the process reflects from the upper boundary immediately upon reaching it. In model (b) the process stays on the upper boundary longer than in model (a). Consequently, the mode in model (c) has higher density than the modes in model (a) or (b) and the mode in model (a) has higher density than the mode in model (b).

Example 2: Consider the scenario in Example 1, but let $\mathbf{T}$ be given instead by

$$\mathbf{T} = \begin{bmatrix} -1.1 & 1.1 \\ 1 & -1 \end{bmatrix}.$$

The unbounded process $(M(t), \varphi(t))$ is again transient — this time with a downward drift [10]. We construct two different bounded processes $(\widetilde{M}(t), \varphi(t))$ with parameters

(a) $\hat{\mathbf{P}} = [1, 0]$, $\hat{\mathbf{T}} = [-0.01, 0.01]$ (absorbing)
(b) $\hat{\mathbf{P}} = [0, 1]$ (immediate reflection)

The results are given in Figures 7 and 8. Note again the significant difference for the probability densities $\psi(t)_{12}$ between the bounded and the unbounded models, even though the unbounded process has a downward drift. The probability densities for the bounded models (a) and (b) vary in the expected way as well.

Example 3: Let

$$\mathbf{T} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

The unbounded process $(M(t), \varphi(t))$ is null-recurrent, with no drift. As discussed after Theorem 2 in Section 3, this raises the possibility of not being able to calculate certain quantities. However, as we wish to numerically invert the Laplace–Stieltjes transforms to determine a probability density function, this is not a problem, as we

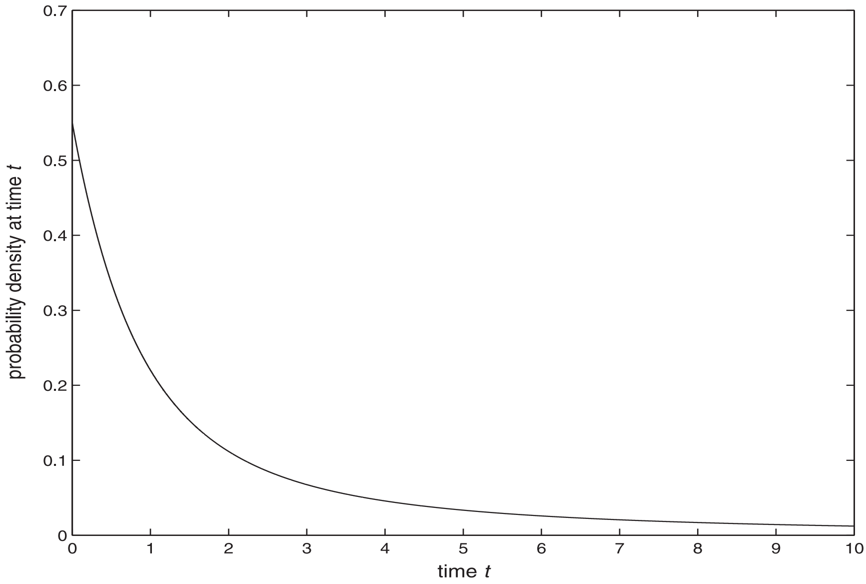

FIGURE 7. Probability density $\psi(t)_{12}$ for the unbounded model in Example 2.

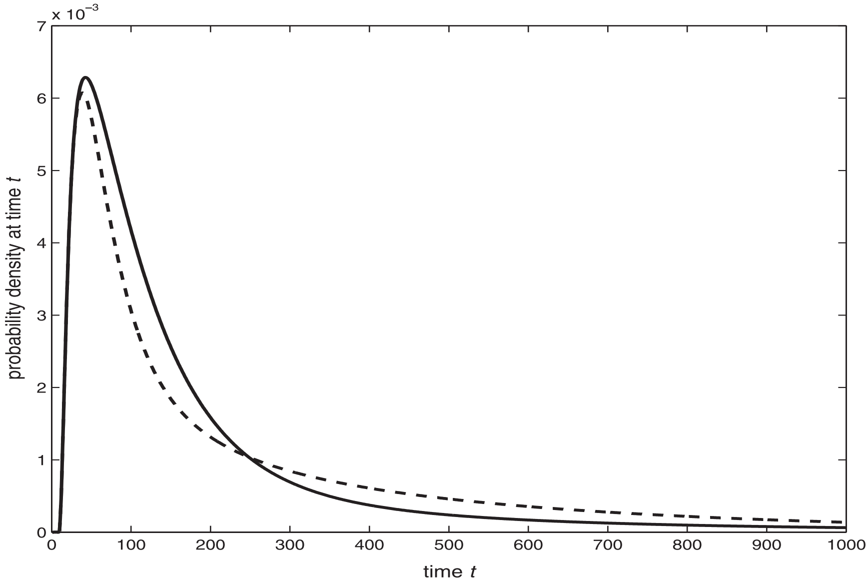

FIGURE 8. Probability densities $\psi(t)_{12}$ for the bounded models in Example 2: (a) dashed line, (b) solid line.

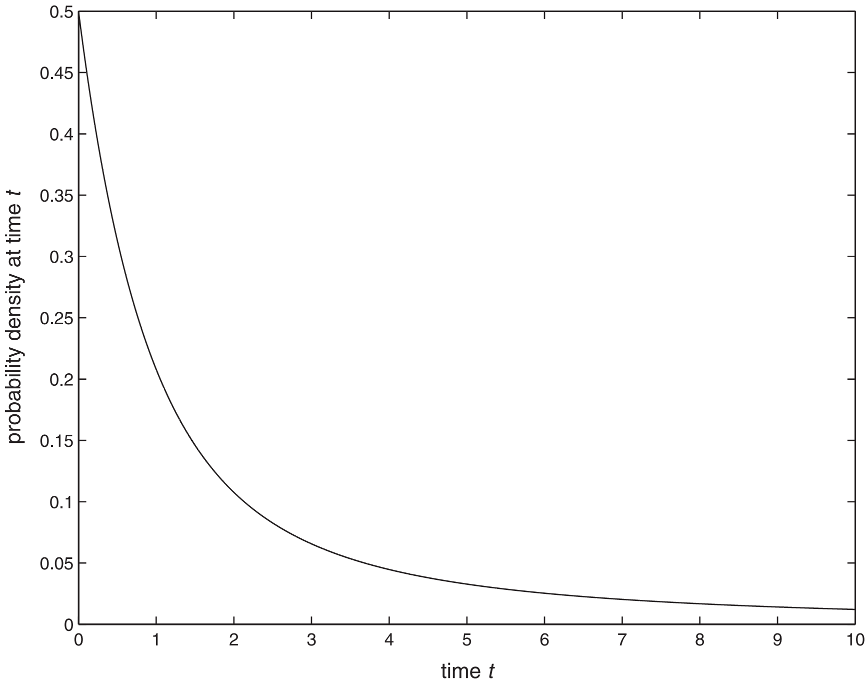

FIGURE 9. Probability density $\psi(t)_{12}$ for the unbounded model in Example 3.

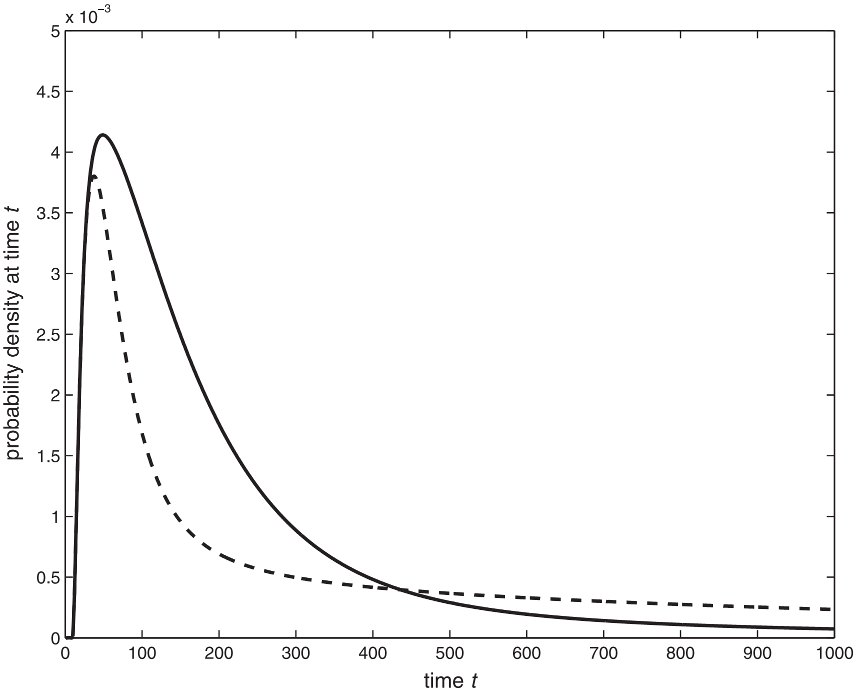

FIGURE 10. Probability densities $\psi(t)_{12}$ for the bounded models in Example 3: (a) dashed line, (b) solid line.

never calculate the transform at $s = 0$. We construct two different bounded processes $(\widetilde{M}(t), \varphi(t))$ with parameters

(a) $\hat{\mathbf{P}} = [1, 0]$, $\hat{\mathbf{T}} = [-0.01, 0.01]$

(b) $\hat{\mathbf{P}} = [0, 1]$

The results are given in Figures 9 and 10. Yet again, the significant difference for the probability densities $\psi(t)_{12}$ for the bounded and the unbounded models is clearly visible. The probability densities for the bounded models (a) and (b) vary more significantly as well.

8. CONCLUSION

We have considered a fluid buffer with an upper and lower boundary. In order to model this buffer, we have constructed a Markovian stochastic fluid flow model in which the fluid level has a lower bound zero and a positive upper bound and in which the behavior of the process at the boundaries is modeled by a number of parameters.

We have established expressions for several performance measures for this model, all of which correspond to the time taken to traverse sample paths. These include formulas for hitting times on the initial level, draining/filling times, and sojourn times in specified sets. Via simple examples, we have shown that we can calculate the probability densities of the required performance measures for all generators, regardless of whether they represent transient or null-recurrent unbounded processes. We have also illustrated the fact that the parameters of the bounded model, particularly at the boundaries, allow for modeling a range of behaviors.

All of the results have been obtained by using techniques applied within the fluid flow model, and useful physical interpretations have been given. We have also considered the unbounded model and extended our previous results in [10] to obtain preliminary results, essential in the treatment of the bounded model.

We extend these ideas in our treatment of a multilayer Markovian fluid model with a wide range of barrier behaviors in [8].

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References

1. Abate, J. & Whitt, W. (1995). Numerical inversion of Laplace transforms of probability distributions. *ORSA Journal on Computing* 7(1): 36–43.
2. Ahn, S., Badescu, A. L. & Ramaswami, V. (2007). Time dependent analysis of finite buffer fluid flows and risk models with a divident barrier. *Queueing Systems* 55: 207–222.
3. Ahn, S., Jeon, J., & Ramaswami, V. (2005). Steady state analysis of finite fluid flow models using finite QBDs. *Queueing Systems* 49(3–4): 223–259.
4. Ahn, S. & Ramaswami, V. (2004). Transient analysis of fluid flow models via stochastic coupling to a queue. *Stochastic Models* 20(1): 71–104.
5. Ahn, S. & Ramaswami, V. (2005). Efficient algorithms for transient analysis of stochastic fluid flow models. *Journal of Applied Probability* 42(2): 531–549.
6. Anick, D., Mitra, D., & Sondhi, M.M. (1982). Stochastic theory of data handling system with multiple sources. *Bell System Technical Journal* 61: 1871–1894.
7. Asmussen, S. (1995). Stationary distributions for fluid flow models with or without Brownian noise. *Stochastic Models* 11: 1–20.
8. Bean, N.G. & O'Reilly, M.M. (2008). Performance measures of a multi-layer Markovian fluid model. *Annals of Operations Research* 160: 99–120.
9. Bean, N.G., O'Reilly, M.M., & Sargison, J. Stochastic fluid flows in the operation and maintenance of hydro-power generators. In preparation.
10. Bean, N.G., O'Reilly, M.M., & Taylor, P.G. (2005). Hitting probabilities and hitting times for stochastic fluid flows. *Stochastic Processes and Their Applications* 115: 1530–1556.
11. Bean, N.G., O'Reilly, M.M., & Taylor, P.G. (2005). Algorithms for return probabilities for stochastic fluid flows. *Stochastic Models* 21(1): 149–184.
12. Bean, N.G., O'Reilly, M.M., & Taylor, P.G. (2008). Algorithms for the Laplace–Stieltjes transforms of the first return probabilities for stochastic fluid flows. *Methodology & Computing in Applied Probability* 10: 381–408.

13. Elwalid, A.I. & Mitra, D. (1991). Analysis and design of rate-based congestion control of high-speed networks, I: Stochastic fluid models, access regulation. *Queueing Systems: Theory and Applications* 9: 19–64.
14. Fulks, W. (1978). *Advanced calculus. Introduction to analysis. 3rd ed.* New York: Wiley.
15. Helfgott, A.E.R., Bean, N.G., Connolly, S., Baird, A., & O'Reilly, M.M. Modelling the resilience of coral reefs to global climate change: A stochastic fluid model of the adaptive bleaching hypothesis on the Great Barrier Reef. In preparation.
16. Mitra, D. (1988). Stochastic theory of a fluid model of procedures and consumers coupled by a buffer. *Advances in Applied Probability* 20: 646–676.
17. Ramaswami, V. (1999). Matrix analytic methods for stochastic fluid flows. In *Proceedings of the 16th International Teletraffic Congress*, pp. 1019–1030.
18. Ramaswami, V. (2006). Passage times in fluid models with application to risk processes. *Methodology and Computing in Applied Probability* 8: 497–515.
19. Remiche, M.-A. (2005). Compliance of the token bucket model for Markovian type traffic. *Stochastic Models* 21(2–3): 615–630.
20. Rogers, L.C. (1994). Fluid models in queueing theory and Wiener–Hopf factorization of Markov chains. *The Annals of Applied Probability* 4(2): 390–413.
21. Silva Soares, A. da & Latouche, G. (2002). Further results on the similarity between fluid queues and QBDs. In G. Latouche & P. Taylor (eds.), *Matrix-analytic methods: Theory and applications.* Singapore: World Scientific, pp. 89–106.
22. Silva Soares, A. da & Latouche, G. (2006). Matrix-analytic methods for fluid queues with finite buffers. *Performance Evaluation* 63: 295–314.
23. Simonian, A. & Virtamo, J. (1991). Transient and stationary distributions for fluid queues and input processes with a density. *SIAM Journal on Applied Mathematics* 51: 1732–1739.
24. Stern, T.E. & Elwalid, A.I. (1991). Analysis of separable Markov-modulated rate models for information-handling systems. *Advances in Applied Probability* 23: 105–139.

APPENDIX A
Proof of Theorem 1

We apply the method used in the proof of [10, Lemma 1] and prove the formula for $\hat{\mathbf{H}}^x(s)$. As in [10], the formula for $\hat{\mathbf{G}}^x(s)$ follows by symmetry.

By using an argument similar to the one used in the proof of [10, Lemma 1], we have that $\hat{\mathbf{H}}^y_{11}(s)$ must be of the form $e^{\mathbf{A}(s)y}$, where the generator $\mathbf{A}(s)$ is defined by

$$\mathbf{A}(s) = \frac{d}{dx}\hat{\mathbf{H}}^x_{11}(s)\bigg|_{x=0}. \tag{A1}$$

Assume that the process $(M(t), \varphi(t))$ starts from level 0 in phase $i \in \mathcal{S}_1$ and is observed until it first reaches level $x > 0$. The only events that occur with probability greater than $o(x)$ are the following:

1. The phase process either remains in phase i until the fluid level moves from 0 to x or makes a single transition from i to some phase $j \neq i \in \mathcal{S}_1$ at some level u in $(0, x]$ and remains in phase j until the fluid level reaches x.

By [10], the corresponding Laplace–Stieltjes transform is the (i,i)th or the (i,j)th entry of an $s_1 \times s_1$ matrix $\hat{\mathbf{H}}_1^x(s)$ such that

$$\frac{d}{dx}\hat{\mathbf{H}}_1^x(s)\bigg|_{x=0} = \mathbf{C}_1^{-1}(\mathbf{T}_{11} - s\mathbf{I}). \tag{A2}$$

2. The phase process makes a transition from i to some phase in $\mathcal{S}_0$ at some level u in $(0,x]$, spends some time in the set $\mathcal{S}_0$, then makes a transition from the set $\mathcal{S}_0$ to some phase $j \in \mathcal{S}_1$, and remains in phase j until the fluid level reaches x.

   By [10], the corresponding Laplace–Stieltjes transform is the (i,j)th entry of an $s_1 \times s_1$ matrix $\hat{\mathbf{H}}_2^x(s)$ such that

$$\frac{d}{dx}\hat{\mathbf{H}}_2^x(s)\bigg|_{x=0} = -\mathbf{C}_1^{-1}\mathbf{T}_{10}(\mathbf{T}_{00} - s\mathbf{I})^{-1}\mathbf{T}_{01}. \tag{A3}$$

3. The phase process
   - makes a single transition from i to some phase $k \in \mathcal{S}_2$ at some level u in $(0,x]$
   - first returns to level u and does so in phase j
   - remains in phase j until the fluid level moves from u to x

   To calculate the Laplace–Stieltjes transform $\hat{\mathbf{H}}_3^x(s)$ of the time taken to do this, observe the following:
   - By the argument in [10], the probability density that the phase process leaves state i at level $u \in (0,x]$, and hence does so at time u/c_i, is $(\lambda_i/c_i)e^{-\lambda_i(u/c_i)}$, where $\lambda_i = -[\mathbf{T}]_{ii}$.
   - The probability that on leaving the phase i, the transition $i \to k$ occurs, where k is some phase in $\mathcal{S}_2$, is given by $[\mathbf{T}]_{ik}/\lambda_i$.
   - The Laplace–Stieltjes transform of the time taken, starting from level u in phase k at time 0, to first return to level u and do so in phase j, is given by $[\hat{\Xi}(s)]_{kj}$.
   - The probability that the phase process remains in phase j as the fluid level moves from u to x, which takes time $(x-u)/c_j$, is given by $e^{-\lambda_j((x-u)/c_j)}$.

   The Laplace–Stieltjes transform corresponding to this case is an $s_1 \times s_1$ matrix such that

$$[\hat{\mathbf{H}}_3^x(s)]_{ij} = \sum_k \int_0^x e^{-s((u/c_i)+(x-u)/c_j)}\frac{1}{c_i}e^{-\lambda_i(u/c_i)}[\mathbf{T}_{12}]_{ik}e^{-\lambda_j((x-u)/c_j)}[\hat{\Xi}(s)]_{kj}\,du.$$

In a manner similar to steps 1–3 in the proof of [10, Lemma 1], we obtain

$$\frac{d}{dx}[\hat{\mathbf{H}}_3^x(s)]_{ij}\bigg|_{x=0} = \frac{1}{c_i}[\mathbf{T}_{12}\hat{\Xi}(s)]_{ij}, \tag{A4}$$

and so

$$\frac{d}{dx}\hat{\mathbf{H}}_3^x(s)\bigg|_{x=0} = \mathbf{C}_1^{-1}\mathbf{T}_{12}\hat{\Xi}(s). \tag{A5}$$

4. The phase process
   - makes a single transition from i to some phase $\ell \in \mathcal{S}_0$ at some level u in $(0, x]$
   - remains in the set $\mathcal{S}_0$ for some finite time, which ends with a transition to some phase $k \in \mathcal{S}_2$
   - first returns to level u and does so in phase j
   - remains in phase j until the fluid level moves from u to x

   We repeat the argument used in step 3, modified by replacing the initial single transition $\mathcal{S}_1 \to \mathcal{S}_2$ (with probability matrix $\mathbf{T}_{12}$) with the sequence $\mathcal{S}_1 \to \mathcal{S}_0 \cdots \mathcal{S}_0 \to \mathcal{S}_2$ (with probability density matrix $\mathbf{T}_{10}e^{\mathbf{T}_{00}t}\mathbf{T}_{02}$). The Laplace–Stieltjes transform corresponding to this case is an $s_1 \times s_1$ matrix such that

$$[\hat{\mathbf{H}}_4^x(s)]_{ij} = \sum_{\ell,k} \int_0^x \int_0^\infty e^{-s((u/c_i)+(x-u/c_j)+t)} \frac{1}{c_i} e^{-\lambda_i(u/c_i)} [\mathbf{T}_{10}]_{i\ell} [e^{\mathbf{T}_{00}t}\mathbf{T}_{02}]_{\ell k} \times e^{-\lambda_j((x-u)/c_j)}\, dt [\hat{\Xi}(s)]_{kj}\, du.$$

   In a manner similar to step 3, we obtain

$$\frac{d}{dx}[\hat{\mathbf{H}}_4^x(s)]_{ij}\Big|_{x=0} = \frac{1}{c_i}[\mathbf{T}_{10} \int_0^\infty e^{-st} e^{\mathbf{T}_{00}t}\, dt \mathbf{T}_{02}\hat{\Xi}(s)]_{ij}, \tag{A6}$$

   and so

$$\frac{d}{dx}\hat{\mathbf{H}}_4^x(s)\Big|_{x=0} = -\mathbf{C}_1^{-1}\mathbf{T}_{10}(\mathbf{T}_{00} - s\mathbf{I})^{-1}\mathbf{T}_{02}\hat{\Xi}(s). \tag{A7}$$

Summarizing, by (A1)–(A7) we have

$$\begin{aligned} A(s) &= \frac{d}{dx}\hat{\mathbf{H}}_{11}^x(s)\Big|_{x=0} \\ &= \frac{d}{dx}(\hat{\mathbf{H}}_1^x(s) + \hat{\mathbf{H}}_2^x(s) + \hat{\mathbf{H}}_3^x(s) + \hat{\mathbf{H}}_4^x(s))\Big|_{x=0} \\ &= \mathbf{C}_1^{-1}[(\mathbf{T}_{11} - s\mathbf{I}) + \mathbf{T}_{12}\hat{\Xi}(s) - \mathbf{T}_{10}(\mathbf{T}_{00} - s\mathbf{I})^{-1}\{\mathbf{T}_{01} + \mathbf{T}_{02}\hat{\Xi}(s)\}]. \end{aligned} \tag{A8}$$

Hence, the formula for $\hat{\mathbf{H}}_{11}^x(s)$ follows.

Next, we observe that in order to reach level x in phase $j \in \mathcal{S}_1$, starting from level 0 in phase $i \in \mathcal{S}_2$, the following must occur:

- The process must first return to level 0 in some phase $i' \in \mathcal{S}_1$. The Laplace–Stieltjes transform of the time taken to do so is $\hat{\Xi}(s)$.
- Then, starting from level 0 in phase $i' \in \mathcal{S}_1$, the process must first reach level x in phase $j \in \mathcal{S}_1$. The corresponding Laplace–Stieltjes transform of this event is $\hat{\mathbf{H}}_{11}^x(s)$.

The matrix $\hat{\mathbf{H}}_{21}^x(s)$ is obtained by multiplying these two matrixes. Finally, we note that, starting from level 0, the process can first reach level $x > 0$ only in some phase in $\mathcal{S}_1$. Hence, the results follows.

APPENDIX B
Proof of Lemma 1

We show that $\lim_{\epsilon\to 0^+} \hat{\mathcal{G}}_{12}^{x,y}(s)^{(\epsilon)} = \hat{\mathcal{G}}_{12}^{x,y}(s)$. The proof for the remaining block matrixes is analogous.

Let $\beta(t)$ be the total number of transitions $\mathcal{S}_1 \to \mathcal{S}_2$ (peaks) that have occurred in the phase process during time interval $[0,t]$. Given a number of peaks i, we define the matrix $\hat{g}(i)^{x,y}(s)$ such that, for $k \in \mathcal{S}_1$ and $\ell \in \mathcal{S}_2$, the (k,ℓ)th entry is given by

$$\begin{aligned}[\hat{g}(i)^{x,y}(s)]_{k\ell} &= E[e^{-s\theta(0)}; \theta(0) < \infty\,, \theta(0) < \theta(y), \varphi(\theta(0)) \\ &= \ell\,, \beta(\theta(0)) = i \mid M(0) = x, \varphi(0) = k]. \end{aligned} \quad \textbf{(B1)}$$

The physical interpretation of $[\hat{g}(i)^{x,y}(s)]_{k\ell}$ is similar to $[\hat{\mathcal{G}}_{12}^{x,y}(s)]_{k\ell}$, with an additional condition that there are exactly i peaks in a sample path contributing to $[\hat{\mathcal{G}}_{12}^{x,y}(s)]_{k\ell}$ for the process $(M(t), \varphi(t))$. We therefore have

$$\hat{\mathcal{G}}_{12}^{x,y}(s) = \sum_{i=1}^{\infty} \hat{g}(i)^{x,y}(s). \quad \textbf{(B2)}$$

In a similar manner, we define $\hat{g}(i)^{x,y}(s)^{(\epsilon)}$ for each process $(M(t)^{(\epsilon)}, \varphi(t)^{(\epsilon)})$, so that

$$\hat{\mathcal{G}}_{12}^{x,y}(s)^{(\epsilon)} = \sum_{i=1}^{\infty} \hat{g}(i)^{x,y}(s)^{(\epsilon)}. \quad \textbf{(B3)}$$

Following [10], let

$$\begin{aligned} Q_{11}(s) &= C_1^{-1}[(T_{11} - sI) - T_{10}(T_{00} - sI)^{-1}T_{01}], \\ Q_{22}(s) &= C_2^{-1}[(T_{22} - sI) - T_{20}(T_{00} - sI)^{-1}T_{02}], \\ Q_{12}(s) &= C_1^{-1}[T_{12} - T_{10}(T_{00} - sI)^{-1}T_{02}], \\ Q_{21}(s) &= C_1^{-1}[T_{21} - T_{20}(T_{00} - sI)^{-1}T_{01}]. \end{aligned}$$

In a similar way, we define $Q_{11}(s)^{(\epsilon)}$, $Q_{12}(s)^{(\epsilon)}$, $Q_{21}(s)^{(\epsilon)}$, and $Q_{22}(s)^{(\epsilon)}$ for each process $(M(t)^{(\epsilon)}, \varphi(t)^{(\epsilon)})$. The physical interpretations of the above matrixes are given in [10].

By conditioning on the first level of decrease, we have

$$\hat{g}(1)^{x,y}(s) = \int_{u=0}^{y-x} e^{Q_{11}(s)u} Q_{12}(s) e^{Q_{22}(s)(u+x)}\, du, \quad \textbf{(B4)}$$

$$\hat{g}(1)^{x,y}(s)^{(\epsilon)} = \int_{u=0}^{y-x} e^{Q_{11}(s)^{(\epsilon)}u} Q_{12}(s)^{(\epsilon)} e^{Q_{22}(s)^{(\epsilon)}(u+x)}\, du, \quad \textbf{(B5)}$$

and

$$\hat{g}(i+1)^{x,y}(s) = \int_{u=0}^{y-x} e^{Q_{11}(s)u} Q_{12}(s) \int_{w=0}^{u+x} e^{Q_{22}(s)w} Q_{21}(s)\hat{g}(i)^{x+u-w,y}(s)\, dw\, du, \quad \textbf{(B6)}$$

$$\hat{g}(i+1)^{x,y}(s)^{(\epsilon)} = \int_{u=0}^{y-x} e^{Q_{11}(s)^{(\epsilon)}u} Q_{12}(s)^{(\epsilon)} \int_{w=0}^{u+x} e^{Q_{22}(s)^{(\epsilon)}w} Q_{21}(s)^{(\epsilon)}\hat{g}(i)^{x+u-w,y}(s)^{(\epsilon)}\, dw\, du. \quad \textbf{(B7)}$$

The uniform convergence of $\mathcal{T}^{(\epsilon)}$ implies by [14, Thm. 14.3c] that, for all $i \geq 1$,

$$\lim_{\epsilon \to 0^+} \hat{g}(i)^{x,y}(s)^{(\epsilon)} = \hat{g}(i)^{x,y}(s). \quad \textbf{(B8)}$$

Let $S_m^{(\epsilon)}(s) = \sum_{i=1}^{m} \hat{g}(i)^{x,y}(s)^{(\epsilon)}$. The sequence $\{S_m^{(\epsilon)}(s)\}$, $m \geq 1$, is nondecreasing and convergent to $\hat{\mathcal{G}}_{12}^{x,y}(s)^{(\epsilon)}$ as $m \to \infty$ and, hence, by [14, Thm. 14.5], uniformly convergent to $\hat{\mathcal{G}}_{12}^{x,y}(s)^{(\epsilon)}$. Consequently, by [14, Sect. 14.3],

$$\lim_{\epsilon \to 0^+} \lim_{m \to \infty} S_m^{(\epsilon)}(s) = \lim_{m \to \infty} \lim_{\epsilon \to 0^+} S_m^{(\epsilon)}(s), \quad \textbf{(B9)}$$

and so

$$\lim_{\epsilon \to 0^+} \hat{\mathcal{G}}_{12}^{x,y}(s)^{(\epsilon)} = \hat{\mathcal{G}}_{12}^{x,y}(s). \quad \textbf{(B10)}$$