The self-coupled Einstein-Cartan-Dirac equations in terms of Dirac bilinears

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Abstract. In this article we present the algebraic rearrangement, or matrix inversion of the Dirac equation in a curved Riemann-Cartan spacetime with torsion; the presence of non-vanishing torsion is implied by the intrinsic spin-1/2 of the Dirac field. We then demonstrate how the inversion leads to a reformulation of the fully non-linear and self-interactive Einstein-Cartan-Dirac field equations in terms of Dirac bilinears. It has been known for some decades that the Dirac equation for charged fermions interacting with an electromagnetic field can be algebraically inverted, so as to obtain an explicit rational expression of the four-vector potential of the gauge field in terms of the spinors. Substitution of this expression into Maxwell's equations yields the bilinear form of the self-interactive Maxwell-Dirac equations. In the present (purely gravitational) case, the inversion process yields *two* rational four-vector expressions in terms of Dirac bilinears, which are gravitational analogues of the electromagnetic vector potential. These potentials also appear as irreducible summand components of the connection, along with a traceless residual term of mixed symmetry. When taking the torsion field equation into account, the residual term can be written as a function of the object of anholonomity. Using the local tetrad frame associated with observers co-moving with the Dirac matter, a generic vierbein frame can described in terms of four Dirac bilinear vector fields, normalized by a scalar and pseudoscalar field. A corollary of this is that in regions where the Dirac field is non-vanishing, the self-coupled Einstein-Cartan-Dirac equations can in principle be expressed in terms of Dirac bilinears only.

1. Introduction

The Dirac equation, the relativistic wave equation for spin-1/2 fermions, can be made to describe particles interacting with a gauge field by replacing the partial derivative with the covariant derivative for the particular field. For a gauge potential of a given form, the Dirac equation may be solved for the spinor field corresponding to the fermion state. One example solution for an electron in an external field is that of the hydrogen atom, where the Dirac equation correctly predicts fine structure as a result of relativistic corrections to the Hamiltonian [1]. However, the external Dirac-Coulomb solution itself does not explain the famous Lamb shift, which requires a consideration of how radiative corrections provided by the Maxwell field affect the energy of the bound electron [2].

An inversion of the Dirac equation can be performed via algebraic rearrangement, such that the gauge potential is written as a rational, explicit function of the spinors [3]. The outcome of this rearrangement procedure appears as though we have performed a matrix inversion, since the Dirac equation can be written in the form

$$MA = R, \tag{1}$$

where the complex 4×4 spinor-vector matrix M is a function only of the components of the Dirac spinor. Assuming the vector potential A is real, M is invertible, and an explicit expression for A in terms of the spinors can be obtained [4]. Substituting the inverted Dirac equation into the equations of motion for the gauge field results in a self-coupled system, where the charged fermion field interacts with itself in an internally consistent way. A central aspect of the algebraic inversion procedure is that the spinors do not appear as stand-alone objects, but rather as *bilinear* combinations. An early proponent of using the bilinear description of Dirac states as the objects of primary interest was Takabayasi [5], who promoted the idea of a relativistic hydrodynamical model of Dirac matter. The states of this model were not spinors or wavefunctions, but *tensors* corresponding to quantum observables, such as current and spin densities. This in effect was an early substantial attempt to formulate a *semi-classical* fluid model of relativistic quantum electrodynamics.

There exists a rich set of interrelationships between quadratic combinations of Dirac bilinears, known as *Fierz identities* [6], [7] (alternatively, Fierz-Pauli-Kofink identities [8]); derived via a successive set of *Fierz expansions* over a Dirac Clifford algebra primitive set of sixteen basis elements. Using a similar process, Crawford showed that [9], given a set of sixteen bilinears formed from this set, the spinor field is recoverable up to a constant spinor with arbitrary phase. In addition to this set, there are two bilinears which are the real and imaginary parts of a complex bilinear (constructed with a both charge conjugated and a regular spinor: $\overline{\psi}^c \gamma^a \psi$), and comprise a locally orthonormal tetrad frame along with the standard Lorentz four-vector and axial-vector fields [10].

In the electromagnetic case, the self-coupled *Maxwell-Dirac* equations were shown to be describable in terms of the gauge *independent* bilinears only, by Inglis and Jarvis [11], manifestly reflecting the physical gauge invariance of the system. Furthermore, these equations were able to be greatly simplified via the applications of infinitesimal invariance under several subgroups of the Poincaré group. These subgroups were chosen from a set of 158 given by Patera, Winternitz & Zassenhaus [12], where a comprehensive list of all the Poincaré Lie subalgebras and their corresponding generators are given. These symmetry reductions aid in the search for solutions to an otherwise intractable set of non-linear equations.

The ability to invert the Dirac equation is not limited to the electromagnetic case either, and we showed in a previous publication [13] that an inversion can be performed for the non-Abelian gauge field $SU(2)$. We found that the algebraic process was very similar to the Abelian case, but with some extra difficulty, and the inverted form was given implicitly. It is currently unclear whether a similar generalisation exists for the strong $SU(3)$ case, although the $SU_L(2) \times U(1)$ electroweak case appears to be promising. Substitution into the

Yang-Mills equations yields a fully non-linear self-interactive non-Abelian hydrodynamical theory, relevant to the study of non-perturbative high-energy plasmas. Another, simpler approach to modelling aspects of non-Abelian hydrodynamics, is to generalize the classical fluid mechanical equations to include local internal symmetries. A description of the non-Abelian Lorentz force involving chromoelectric and chromomagnetic field couplings was obtained this way in [14].

In this paper, we demonstrate how the Dirac equation in a curved Riemann-Cartan spacetime with torsion can be algebraically inverted, in an analogous manner to the $U(1)$ and $SU(2)$ cases; the covariant derivative we use contains the connection contracted with the generator for Lorentz transformations. Due to the intrinsic spin carried by the Dirac fermions, we consider the torsion field generated from the spin current density to be non-vanishing in general [15]; an extra set of constraints on the gravitational field are obtained as a result. In comparison with (1), the curved spacetime Dirac equation is of the form

$$M\Omega + N\Omega_5 = R, \tag{2}$$

and the matrix inversion procedure yields explicit rational expressions for the gravitational "vector potentials" Ω and Ω_5. In section 2, we derive an equation of the form (2) from the standard Dirac equation in curved spacetime. We do this by considering an irreducible decomposition of the connection in $GL(4)$, which can be written as a sum of three terms. The trace term is a function of Ω_a, and the two traceless terms are a fully antisymmetric function of Ω_{5a} and a residual term of mixed symmetry, ${}^{(3)}\Gamma_{abc}$.

In section 3, we give our definition of the tensor fields resulting from sandwiching elements of the Dirac Clifford algebra basis between Dirac spinors. Using this notation, we then show that by left-multiplying the curved spacetime Dirac equation and its charge conjugate with four different spinors, the resulting set of four equations can be solved explicitly for the *two* gravitational vector potentials. These expressions are rational functions of bilinears and their first derivatives, but are not able to be expressed in terms of our tensor field set without further calculations.

The process by which we can write the inverted expressions in terms of bilinear tensor fields is given in 4. Here, we give a brief outline of the process by which Fierz expansions, where an outer product of two Dirac spinors is expanded in the Dirac Clifford algebra basis, are used to derive Fierz identities which are quadratic in the bilinears. These identities are then used to eliminate the explicit appearance of Dirac spinors in the inverted forms of the Dirac equation, replacing them with pure tensor expressions.

Section 5 is given in four parts. In the first two parts, we describe the field equations of the Einstein-Cartan system, for the gravitational dynamics of space-time curvature and spin-torsion respectively. Expressions for the Ricci tensor, scalar, and the torsion are given in terms of the connection and the object of anholonomity. In the third part, we show how the algebraic torsion field equation can be used to place constraints on Ω_a and Ω_{5a}, and to derive an explicit expression for ${}^{(3)}\Gamma_{abc}$ in terms of the object of anholonomity. In the final part, we use the existence of a locally orthonormal tetrad frame corresponding to a family of observers co-moving with the Dirac matter (which arises from the Fierz identities), to generate an expression for the generic vierbein field as a function of the Dirac bilinears.

A summary of our formulation of the self-coupled Einstein-Cartan system is given in Section 6, which demonstrates in principle how this system can be reduced to a set of relations between Dirac bilinears only. A glossary of the symbols we use in this paper are given in Appendix A. Further reduction and analysis of this system is left for future publications.

2. The Einstein-Cartan-Dirac equations and conventions

The Einstein-Cartan-Dirac equations in curved spacetime with torsion have the form

$$(\mathrm{i}\gamma^a e_a{}^\mu(x)\nabla_\mu - m)\psi = 0, \tag{3}$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{4}$$

$$\Upsilon_{\mu\nu}{}^\gamma + \delta^\gamma_\mu \Upsilon_{\nu\sigma}{}^\sigma - \delta^\gamma_\nu \Upsilon_{\mu\sigma}{}^\sigma = 8\pi G \Sigma_{\mu\nu}{}^\gamma. \tag{5}$$

The Dirac equation (3) governs the dynamics of the matter sector of this system; namely, the relativistic wave-like behaviour of spin-1/2 fermionic matter. The gravitational field in the presence of Dirac matter has both curvature and torsion due to the stress-energy and spin of the Dirac matter respectively; the Einstein field and Cartan torsion equations, (4) and (5), describe these relationships.

The focus of this paper is primarily on (3) and its explicit invertibility for irreducible components of the connection. A brief discussion of the Einstein field equation (4) is given in subsection 5.1. The utilization of the torsion field equation (5) to obtain further constraints on the connection is presented in subsections 5.2 and 5.3. Our end result will be an in-principle integration of (3) and (5) with (4), with the ability to express (4) entirely in terms of Dirac bilinears, the state densities of matter which also act as the source of gravity. Deeper analysis of the Einstein equation using the inverted form of the Dirac equation and torsion constraints is beyond the scope of this paper, and is left to future publications.

For Dirac matter, the stress-energy and spin densities are given in terms of the spinors as [16], [17]

$$T_{\mu\nu} = \frac{\mathrm{i}}{2}[\overline{\psi}\gamma_\mu(\nabla_\nu\psi) - (\nabla_\nu\overline{\psi})\gamma_\mu\psi], \tag{6}$$

$$\Sigma_{\mu\nu\gamma} = \frac{\mathrm{i}}{4}\overline{\psi}\gamma_{[\mu}\gamma_\nu\gamma_{\gamma]}\psi. \tag{7}$$

Greek and Latin indices run from 0 to 3, and correspond to coordinate and locally orthonormal frames respectively. The vierbein field $e_a{}^\mu(x)$ relates these two frames locally at each point x, and is quadratically related to the metric, according to

$$g_{\mu\nu}(x) = e^a{}_\mu(x)e^b{}_\nu(x)\eta_{ab}. \tag{8}$$

For the Minkowski spacetime metric we use the particle physics sign convention, whereby the signature is negative in the spatial components:

$$\eta_{ab} := \mathrm{diag}(1, -1, -1, -1). \tag{9}$$

2.1. The gravitational four-vector potentials

For Dirac spinor fields, the covariant derivative is of the form [18], [19]

$$\nabla_\mu \psi = \partial_\mu \psi + \Gamma_\mu \psi, \tag{10}$$

where in the spinor representation, the connection coefficients are

$$\Gamma_\mu = \frac{1}{2}\Gamma_\mu{}^{ab} S_{ab} = -\frac{\mathrm{i}}{2}\Gamma_\mu{}^{ab}\sigma_{ab}. \tag{11}$$

The object $\Gamma_\mu{}^{ab}$ with the leading index in the world-coordinate (holonomic) frame and the other two indices in the local (anholonomic) frame, is often referred to as the *spin connection*, however we shall mostly refer to it as simply the connection. The connection transforms *inhomogeneously* between the coordinate and local frames, according to [20]

$$\Gamma_a{}^{bc} = e_a{}^\mu e^{b\nu} e^c{}_\lambda \Gamma_{\mu\nu}{}^\lambda - e_a{}^\mu e^{b\nu}\partial_\mu e^c{}_\nu, \tag{12}$$

where $\Gamma_a{}^{bc} = e_a{}^\mu \Gamma_\mu{}^{bc}$. Note that because of the intrinsic spin of the Dirac field, $\Gamma_{\mu\nu}{}^\lambda$ is in general *asymmetric* in μ, ν, resulting in a non-vanishing spacetime torsion [21], [15]. The infinitesimal Lorentz generators in the Dirac spinor representation are

$$S_{ab} = -\frac{\mathrm{i}}{2}\sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b], \tag{13}$$

where γ_a are the *Dirac matrices*, and $\sigma_{ab} \equiv \mathrm{i}/2[\gamma_a, \gamma_b]$. Taking account of the Dirac matrix anticommutator

$$\{\gamma_a, \gamma_b\} = 2\eta_{ab}, \tag{14}$$

it can be shown that the right-hand side of (13) satisfies the Lie bracket identity for Lorentz generators

$$[S_{ab}, S_{cd}] = \eta_{ad} S_{bc} + \eta_{bc} S_{ad} - \eta_{ac} S_{bd} - \eta_{bd} S_{ac}. \tag{15}$$

Using (11) and (13), we can rewrite the covariant derivative of the Dirac spinor as

$$\nabla_\mu \psi = \partial_\mu \psi + \frac{1}{8}\Gamma_\mu{}^{ab}[\gamma_a, \gamma_b]\psi. \tag{16}$$

Substituting this into (3), then absorbing the vierbeins and rearranging, the Dirac equation becomes

$$\frac{\mathrm{i}}{8}\Gamma^{abc}\gamma_a[\gamma_b, \gamma_c]\psi = -(\mathrm{i}\gamma^a\partial_a - m)\psi, \tag{17}$$

with $\gamma^a \partial_a \equiv \gamma^a e_a{}^\mu \partial_\mu$. Using the Dirac identity

$$\gamma_a\gamma_b\gamma_c = \eta_{ab}\gamma_c + \eta_{bc}\gamma_a - \eta_{ac}\gamma_b - \mathrm{i}\epsilon_{abcd}\gamma_5\gamma^d, \tag{18}$$

we can write the commutator in the last two indices as

$$\gamma_a[\gamma_b, \gamma_c] = 2(\eta_{ab}\gamma_c - \eta_{ac}\gamma_b - \mathrm{i}\epsilon_{abcd}\gamma_5\gamma^d). \tag{19}$$

The conventions we use for γ_5 and the Levi-Civita symbol are those of Itzykson and Zuber [22]:

$$\epsilon^{abcd} = -\epsilon_{abcd} = \begin{cases} +1 & \text{if } \{a,b,c,d\} \text{ even} \\ -1 & \text{odd} \\ 0 & \text{otherwise,} \end{cases} \tag{20}$$

$$\gamma^5 = \gamma_5 = -(\mathrm{i}/4!)\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = -\mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3. \tag{21}$$

The left-hand side operator of (17) therefore becomes

$$\begin{aligned} \frac{\mathrm{i}}{8}\Gamma^{abc}\gamma_a[\gamma_b,\gamma_c] &= \frac{\mathrm{i}}{4}\Gamma^{abc}(\eta_{ab}\gamma_c - \eta_{ac}\gamma_b - \mathrm{i}\epsilon_{abcd}\gamma_5\gamma^d) \\ &= \frac{\mathrm{i}}{4}(\eta_{ab}\eta_{cd} - \eta_{ac}\eta_{bd})\Gamma^{abc}\gamma^d + \frac{1}{4}\epsilon_{abcd}\Gamma^{abc}\gamma_5\gamma^d \\ &= \Omega_d\gamma^d + \Omega_{5d}\gamma_5\gamma^d, \end{aligned} \tag{22}$$

where we define the *gravitational vector potentials* as

$$\Omega_d := \frac{1}{4}\delta_{adbc}\Gamma^{abc} = \frac{\mathrm{i}}{2}\Gamma_c{}^c{}_d, \tag{23}$$

$$\Omega_{5d} := \frac{1}{4}\epsilon_{abcd}\Gamma^{abc}, \tag{24}$$

with the mixed symmetry imaginary Sylvester tensor [23]

$$\delta_{abcd} := \mathrm{i}(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}), \tag{25}$$

playing a dual role to the Levi-Civita tensor.

2.2. Connection Irreducible Decomposition

From the definitions (23) and (24), we can see that the gravitational vector potentials correspond to components of the connection Γ_{abc}. Now, since the connection corresponds to a rank-3 representation of the local Lorentz group $SO(1,3)$, we can write it as the sum of three irreducible components:

$$\Gamma_{abc} = {}^{(1)}\Gamma_{abc} + {}^{(2)}\Gamma_{abc} + {}^{(3)}\Gamma_{abc}. \tag{26}$$

Due to the antisymmetry of the connection in its second and third indices in the local frame, this irreducible decomposition can equivalently be written in terms of Young patterns as

$$[1] \otimes [11] = [1] \oplus [111] \oplus [21], \tag{27}$$

corresponding to (1) a trace term, (2) a fully antisymmetric term, and (3) a traceless mixed-symmetry term respectively. Written in terms of the connection, the irreducible parts are

$${}^{(1)}\Gamma_{abc} = \frac{1}{3}\eta_{ab}\Gamma_d{}^d{}_c - \frac{1}{3}\eta_{ac}\Gamma_d{}^d{}_b = -\frac{\mathrm{i}}{3}\delta_{aebc}\Gamma_d{}^{de}, \tag{28a}$$

$${}^{(2)}\Gamma_{abc} = \frac{1}{3}(\Gamma_{abc} + \Gamma_{bca} + \Gamma_{cab}), \tag{28b}$$

$${}^{(3)}\Gamma_{abc} = \frac{1}{3}(2\Gamma_{abc} - \Gamma_{bca} - \Gamma_{cab}) + \frac{\mathrm{i}}{3}\delta_{aebc}\Gamma_d{}^{de}. \tag{28c}$$

Using (23) and (24), we can express two of the three irreducible components of the connection in terms of the gravitational four-vector "potentials":

$$^{(1)}\Gamma_{abc} = -\frac{2}{3}\delta_{adbc}\Omega^d, \qquad (29a)$$

$$^{(2)}\Gamma_{abc} = -\frac{2}{3}\epsilon_{abcd}\Omega_5^d, \qquad (29b)$$

The connection can now be written as

$$\Gamma_{abc} = -\frac{2}{3}\delta_{adbc}\Omega^d - \frac{2}{3}\epsilon_{abcd}\Omega_5^d + {}^{(3)}\Gamma_{abc}. \qquad (30)$$

We have thus obtained an expression for the connection in the local frame, which allows for its replacement in terms of the bilinear Dirac matter states via the inverted forms of the Dirac equation (70) and (71), with the exception of the residual term $^{(3)}\Gamma_{abc}$. As we shall see in subsection 5.3, $^{(3)}\Gamma_{abc}$ can be replaced by the irreducible traceless mixed-symmetry component of the object of anholonomity (79), which itself can be replaced by Dirac bilinears when the vierbein is chosen to be the bilinear tetrad (63). Thus, we will be able to obtain an expression for the connection entirely in terms of Dirac bilinears.

2.3. Charge conjugation and comparison with electromagnetism

In terms of the Ω-potentials, the Dirac equation now reads

$$\Omega_a\gamma^a\psi + \Omega_{5a}\gamma_5\gamma^a\psi = -(\mathrm{i}\gamma^a\partial_a - m)\psi. \qquad (31)$$

According to the gauging of the Poincaré group [15], there are four translation-type potentials $\theta^a = e_\mu{}^a dx^\mu$ and six Lorentz-type potentials $\Gamma^{ab} = \Gamma_\mu{}^{ab}dx^\mu$. By introducing the two new four-vector potentials Ω^a and Ω_5^a to replace irreducible parts of the connection, we have increased the number of components from 6 to $4 + 4 = 8$. The inverted Dirac equation provides two explicit expressions for these potentials, (70) and (71), that reduce the number of independent connection components back down to six.

Equation (31) can be compared with the electromagnetically covariant Dirac equation in *flat* spacetime

$$-qA_a\gamma^a\psi = -(\mathrm{i}\gamma^a\partial_a - m)\psi. \qquad (32)$$

We can see that there is an analogy between Ω_a and $-qA_a$, in the sense that these terms are coupled to $\gamma^a\psi$. However, in electromagnetism there is no equivalent potential to Ω_{5a}, say $-qA_{5a}$, which couples to $\gamma_5\gamma^a$. Such an analogous term could in principle arise in an Abelian chiral generalization of the electromagnetic gauge group, such as local $U(1)_L \times U(1)_R$ symmetry. It is of interest to note that if the torsion field equation is taken into account in (31), say by directly substituting the constraint (100), the left-hand side of (31) becomes non-linear in the spinors (via the axial vector term k^a), and the Hehl-Datta equation is obtained [24]. However, as our emphasis is on the explicit *inversion* of the Dirac equation for the Ω and Ω_5 "potentials", we shall leave these objects intact.

In order to proceed with the inversion process, we require the Dirac equation for the charge conjugated spinor. It can be shown [25] that in the absence of electromagnetic fields, this equation has exactly the same form as (3) and (31), such that

$$(\mathrm{i}\gamma^a e_a{}^\mu(x)\nabla_\mu - m)\psi^{\mathrm{c}} = 0, \tag{33}$$

and therefore

$$\Omega_a\gamma^a\psi^{\mathrm{c}} + \Omega_{5a}\gamma_5\gamma^a\psi^{\mathrm{c}} = -(\mathrm{i}\gamma^a\partial_a - m)\psi^{\mathrm{c}}. \tag{34}$$

Incidentally, in the electromagnetic case, the sign of the term carrying the charge coupling constant q changes sign under a charge conjugation:

$$+qA_a\gamma^a\psi^{\mathrm{c}} = -(\mathrm{i}\gamma^a\partial_a - m)\psi^{\mathrm{c}}. \tag{35}$$

3. The inversion procedure

The inversion of the Dirac equation for the components of the connection which couple to spin-1/2 fermions proceeds in a similar fashion to the analogous $U(1)$ electromagnetic [11] and non-Abelian $SU(2)$ [13] cases. In all of these cases, the procedure involves the formation of spinor bilinears, which in the tradition of Takabayasi [5], Zhelnorovich [26], and Halbwachs [27], we can write as a set of 16 tensor fields: scalar, pseudoscalar, four-vector, axial four-vector, and rank-2 tensor

$$\sigma = \overline{\psi}\psi, \tag{36a}$$

$$\omega = \overline{\psi}\gamma_5\psi, \tag{36b}$$

$$j^a = \overline{\psi}\gamma^a\psi, \tag{36c}$$

$$k^a = \overline{\psi}\gamma_5\gamma^a\psi, \tag{36d}$$

$$s^{ab} = \overline{\psi}\sigma^{ab}\psi. \tag{36e}$$

In addition, we also have the dual rank-2 tensor

$${}^*s^{ab} = \frac{\mathrm{i}}{2}\epsilon^{abcd}s_{cd} = \overline{\psi}\gamma_5\sigma^{ab}\psi, \tag{37}$$

as well as two four-vectors comprising real and imaginary parts of a complex bilinear

$$m^a + \mathrm{i}n^a = \overline{\psi}^{\mathrm{c}}\gamma^a\psi \tag{38a}$$

$$m^a = \mathrm{Re}\{\overline{\psi}^{\mathrm{c}}\gamma^a\psi\} = \frac{1}{2}(\overline{\psi}^{\mathrm{c}}\gamma^a\psi + \overline{\psi}\gamma^a\psi^{\mathrm{c}}) \tag{38b}$$

$$n^a = \mathrm{Im}\{\overline{\psi}^{\mathrm{c}}\gamma^a\psi\} = \frac{\mathrm{i}}{2}(\overline{\psi}\gamma^a\psi^{\mathrm{c}} - \overline{\psi}^{\mathrm{c}}\gamma^a\psi), \tag{38c}$$

where $\overline{\psi}$ and ψ^{c} are the Dirac and charge conjugate spinors respectively. The bilinear set are all *real*, except for ω and ${}^*s^{ab}$, which are pure imaginary; this is just a choice of convention, which can be altered by defining the new *real* bilinears $-\mathrm{i}\omega$ and $-\mathrm{i}{}^*s^{ab}$. Now, left-multiplying (31) by $\overline{\psi}\gamma^b$ gives

$$\Omega_a\overline{\psi}\gamma^b\gamma^a\psi + \Omega_{5a}\overline{\psi}\gamma^b\gamma_5\gamma^a\psi = -\mathrm{i}\overline{\psi}\gamma^b\gamma^a(\partial_a\psi) + m\overline{\psi}\gamma^b\psi. \tag{39}$$

Applying the Dirac identities

$$\gamma^b\gamma^a = \eta^{ba} - \mathrm{i}\sigma^{ba}, \tag{40a}$$

$$\{\gamma_5, \gamma^a\} = 0, \tag{40b}$$

and writing closed-form bilinears in tensor notation, we get

$$(\sigma\eta^{ba} - \mathrm{i}s^{ba})\Omega_a + (-\omega\eta^{ba} + \mathrm{i}^*s^{ba})\Omega_{5a} = -\mathrm{i}\overline{\psi}(\partial^b\psi) - \overline{\psi}\sigma^{ba}(\partial_a\psi) + mj^b. \tag{41}$$

Likewise, left-multiplying (31) by $\overline{\psi}\gamma_5\gamma^b$, and applying the same Dirac identities yields

$$(\omega\eta^{ba} - \mathrm{i}^*s^{ba})\Omega_a + (-\sigma\eta^{ba} + \mathrm{i}s^{ba})\Omega_{5a} = -\mathrm{i}\overline{\psi}\gamma_5(\partial^b\psi) - \overline{\psi}\gamma_5\sigma^{ba}(\partial_a\psi) + mk^b. \tag{42}$$

Following the same steps with the charge conjugate Dirac equation, left-multiplying 34 by $\overline{\psi}^{\mathrm{c}}\gamma^b$ and $\overline{\psi}^{\mathrm{c}}\gamma_5\gamma^b$ yields the respective equations

$$\begin{aligned}(\overline{\psi}^{\mathrm{c}}\psi^{\mathrm{c}}\eta^{ba} - \mathrm{i}\overline{\psi}^{\mathrm{c}}\sigma^{ba}\psi^{\mathrm{c}})\Omega_a + (-\overline{\psi}^{\mathrm{c}}\gamma_5\psi^{\mathrm{c}}\eta^{ba} + \mathrm{i}\overline{\psi}^{\mathrm{c}}\gamma_5\sigma^{ba}\psi^{\mathrm{c}})\Omega_{5a} \\ = -\mathrm{i}\overline{\psi}^{\mathrm{c}}(\partial^b\psi^{\mathrm{c}}) - \overline{\psi}^{\mathrm{c}}\sigma^{ba}(\partial_a\psi^{\mathrm{c}}) + m\overline{\psi}^{\mathrm{c}}\gamma^b\psi^{\mathrm{c}},\end{aligned} \tag{43}$$

$$\begin{aligned}(\overline{\psi}^{\mathrm{c}}\gamma_5\psi^{\mathrm{c}}\eta^{ba} - \mathrm{i}\overline{\psi}^{\mathrm{c}}\gamma_5\sigma^{ba}\psi^{\mathrm{c}})\Omega_a + (-\overline{\psi}^{\mathrm{c}}\psi^{\mathrm{c}}\eta^{ba} + \mathrm{i}\overline{\psi}^{\mathrm{c}}\sigma^{ba}\psi^{\mathrm{c}})\Omega_{5a} \\ = -\mathrm{i}\overline{\psi}^{\mathrm{c}}\gamma_5(\partial^b\psi^{\mathrm{c}}) - \overline{\psi}^{\mathrm{c}}\gamma_5\sigma^{ba}(\partial_a\psi^{\mathrm{c}}) + m\overline{\psi}^{\mathrm{c}}\gamma_5\gamma^b\psi^{\mathrm{c}}.\end{aligned} \tag{44}$$

Using the definition for the charge conjugate spinor

$$\psi^{\mathrm{c}} = C\overline{\psi}^{\mathrm{T}} = \mathrm{i}\gamma^2\gamma^0\overline{\psi}^{\mathrm{T}}, \tag{45}$$

we can derive a relationship between bilinears with *non-Grassmann* charge conjugate spinors and regular spinors‡

$$\overline{\psi}^{\mathrm{c}}\Gamma\chi^{\mathrm{c}} = -\overline{\chi}C^{-1}\Gamma^{\mathrm{T}}C\psi, \tag{46}$$

where the spinor χ may have tensor indices (ie. $\chi = \partial_a\psi$), and Γ is an element of the same Dirac-Clifford algebra defining the set (36a)-(36e). Applying the Dirac matrix charge conjugation identities [22]

$$C^{-1}\gamma^{a\mathrm{T}}C = -\gamma^a, \tag{47a}$$

$$C^{-1}\gamma_5^{\mathrm{T}}C = \gamma_5, \tag{47b}$$

$$C^{-1}(\gamma_5\gamma^a)^{\mathrm{T}}C = \gamma_5\gamma^a, \tag{47c}$$

$$C^{-1}\sigma^{ab\mathrm{T}}C = -\sigma^{ab}, \tag{47d}$$

$$C^{-1}(\gamma_5\sigma^{ab})^{\mathrm{T}}C = -\gamma_5\sigma^{ab}, \tag{47e}$$

we can rewrite (43) and (44) as

$$(-\sigma\eta^{ba} - \mathrm{i}s^{ba})\Omega_a + (\omega\eta^{ba} + \mathrm{i}^*s^{ba})\Omega_{5a} = \mathrm{i}(\partial^b\overline{\psi})\psi - (\partial_a\overline{\psi})\sigma^{ba}\psi + mj^b, \tag{48}$$

$$(-\omega\eta^{ba} - \mathrm{i}^*s^{ba})\Omega_a + (\sigma\eta^{ba} + \mathrm{i}s^{ba})\Omega_{5a} = \mathrm{i}(\partial^b\overline{\psi})\gamma_5\psi - (\partial_a\overline{\psi})\gamma_5\sigma^{ba}\psi - mk^b \tag{49}$$

respectively. Subtracting (48) from (41), and (49) from (48), yields the respective equations

$$2\sigma\Omega^a - 2\omega\Omega_5^a = -\mathrm{i}\partial^a\sigma - [\overline{\psi}\sigma^{ab}(\partial_b\psi) - (\partial_b\overline{\psi})\sigma^{ab}\psi], \tag{50}$$

$$2\omega\Omega^a - 2\sigma\Omega_5^a = -\mathrm{i}\partial^a\omega - [\overline{\psi}\gamma_5\sigma^{ab}(\partial_b\psi) - (\partial_b\overline{\psi})\gamma_5\sigma^{ab}\psi] + 2mk^a, \tag{51}$$

‡ In the present work we assume the spinor quantities are c-numbers.

where we have relabelled the indices. Multiplying (50) and (51) by (σ,ω) and (ω,σ) respectively, then subtracting the second equation from the first gives

$$\Omega^a = \frac{1}{2}(\sigma^2-\omega^2)^{-1}\{\mathrm{i}[\omega(\partial^a\omega)-\sigma(\partial^a\sigma)]+\omega[\overline{\psi}\gamma_5\sigma^{ab}(\partial_b\psi)-(\partial_b\overline{\psi})\gamma_5\sigma^{ab}\psi] \\ -\sigma[\overline{\psi}\sigma^{ab}(\partial_b\psi)-(\partial_b\overline{\psi})\sigma^{ab}\psi]-2m\omega k^a\}, \quad (52)$$

$$\Omega_5^a = \frac{1}{2}(\sigma^2-\omega^2)^{-1}\{\mathrm{i}[\sigma(\partial^a\omega)-\omega(\partial^a\sigma)]+\sigma[\overline{\psi}\gamma_5\sigma^{ab}(\partial_b\psi)-(\partial_b\overline{\psi})\gamma_5\sigma^{ab}\psi] \\ -\omega[\overline{\psi}\sigma^{ab}(\partial_b\psi)-(\partial_b\overline{\psi})\sigma^{ab}\psi]-2m\sigma k^a\}, \quad (53)$$

the inverted form of the Dirac equation in curved spacetime.

4. Bilinear refinement using Fierz identities

It is apparent however, that the bracketed second and third terms in (52) and (53) are not closed-form bilinears, due to the minus sign preventing a simple application of the Leibniz rule for derivatives. It is possible to show through a very lengthy algebraic process that Fierz expansions can be used to re-write these terms in closed tensor form. Due to the sheer length and tediousness of these calculations, they are not given here, however their derivation follows a similar process to Appendix C in [11] and Appendix B in [28].

The Fierz expansion can be used to write the outer product of two spinors $\psi\overline{\chi}$, which is a 4×4 matrix in the spinor degrees of freedom, as a sum of terms over the basis of Dirac-Clifford matrices with bilinear coefficients

$$\psi\overline{\chi} = \frac{1}{4}(\overline{\chi}\psi)I+\frac{1}{4}(\overline{\chi}\gamma_a\psi)\gamma^a+\frac{1}{8}(\overline{\chi}\sigma_{ab}\psi)\sigma^{ab}-\frac{1}{4}(\overline{\chi}\gamma_5\gamma_a\psi)\gamma_5\gamma^a+\frac{1}{4}(\overline{\chi}\gamma_5\psi)\gamma_5, \quad (54)$$

which can be derived from the more formal expression

$$\psi\overline{\chi} = \sum_{R=1}^{16} a_R\Gamma_R \quad (55)$$

where $R=1,...,16$ runs over all of the elements of the Dirac-Clifford algebra. Multiplying (55) from the right by Dirac matrix Γ_B [where B runs over the types: *scalar*, ..., *rank-2 tensor* in (36a)-(36e)], and using the trace identities

$$\mathrm{Tr}(\Gamma_R\Gamma_B) = \begin{cases} \mathrm{Tr}(\Gamma_B^2) & \text{if } R=B, \\ 0 & \text{otherwise,} \end{cases} \quad (56)$$

$$\mathrm{Tr}(\psi\overline{\chi}\Gamma_B) = \overline{\chi}\Gamma_B\psi, \quad (57)$$

along with the trace properties of the Dirac matrices, one can easily derive (54).

Following a very tedious process of applying (54) to the terms in (52) and (53) where the spinors are visible, we obtain the purely bilinear expressions

$$\omega[\overline{\psi}\gamma_5\sigma^{ab}(\partial_b\psi)-(\partial_b\overline{\psi})\gamma_5\sigma^{ab}\psi]-\sigma[\overline{\psi}\sigma^{ab}(\partial_b\psi)-(\partial_b\overline{\psi})\sigma^{ab}\psi] \\ =(\sigma^2-\omega^2)^{-1}\{s^{ab}[\omega j^c(\partial_b k_c)+\mathrm{i}\sigma m^c(\partial_b n_c)]-{}^*s^{ab}[\sigma j^c(\partial_b k_c)+\mathrm{i}\omega m^c(\partial_b n_c)]\} \\ +\delta^{abcd}[k_c(\partial_b k_d)-j_c(\partial_b j_d)], \quad (58)$$

$$\sigma[\overline{\psi}\gamma_5\sigma^{ab}(\partial_b\psi) - (\partial_b\overline{\psi})\gamma_5\sigma^{ab}\psi] - \omega[\overline{\psi}\sigma^{ab}(\partial_b\psi) - (\partial_b\overline{\psi})\sigma^{ab}\psi]$$
$$= (\sigma^2-\omega^2)^{-1}\{s^{ab}[\sigma j^c(\partial_b k_c) + \mathrm{i}\omega m^c(\partial_b n_c)] - {}^*s^{ab}[\omega j^c(\partial_b k_c) + \mathrm{i}\sigma m^c(\partial_b n_c)]\}$$
$$+ \epsilon^{abcd}[k_c(\partial_b k_d) - j_c(\partial_b j_d)]. \tag{59}$$

Using the Fierz identities [9]

$$s^{ab} = (\sigma^2-\omega^2)^{-1}(\sigma\epsilon^{abcd} - \omega\delta^{abcd})j_c k_d \tag{60a}$$
$${}^*s^{ab} = (\sigma^2-\omega^2)^{-1}(\omega\epsilon^{abcd} - \sigma\delta^{abcd})j_c k_d, \tag{60b}$$
$$\mathrm{i}\epsilon^{abcd}j_c k_d = \mathrm{i}(m^a n^b - m^b n^a) = \delta^{abcd}m_c n_d, \tag{60c}$$
$$\mathrm{i}\delta^{abcd}j_c k_d = -j^a k^b + j^b k^a = \epsilon^{abcd}m_c n_d, \tag{60d}$$

the expressions within the curved braces can be recast as

$$s^{ab}[\omega j^c(\partial_b k_c) + \mathrm{i}\sigma m^c(\partial_b n_c)] - {}^*s^{ab}[\sigma j^c(\partial_b k_c) + \mathrm{i}\omega m^c(\partial_b n_c)]$$
$$= \delta^{abcd}[j_c j^e k_d(\partial_b k_e) + m_c m^e n_d(\partial_b n_e)], \tag{61}$$
$$s^{ab}[\sigma j^c(\partial_b k_c) + \mathrm{i}\omega m^c(\partial_b n_c)] - {}^*s^{ab}[\omega j^c(\partial_b k_c) + \mathrm{i}\sigma m^c(\partial_b n_c)]$$
$$= \epsilon^{abcd}[j_c j^e k_d(\partial_b k_e) + m_c m^e n_d(\partial_b n_e)]. \tag{62}$$

To proceed further, we require the tetrad frame of four-vector bilinears, with scalar normalizing factor:

$$t_\alpha{}^a = (\sigma^2-\omega^2)^{-1/2}[j^a, m^a, n^a, k^a], \tag{63}$$

where $\alpha = 0,1,2,3$ labels the columns. The details of this local frame are discussed in subsection 5.4. The tetrad orthonormality implies

$$t_\alpha{}^a t^\alpha{}_b = \delta^a{}_b = (\sigma^2-\omega^2)^{-1}(j^a j_b - m^a m_b - n^a n_b - k^a k_b), \tag{64}$$

and taking the derivative yields

$$t_\alpha{}^a(\partial_b t_{\beta a}) = -t_\beta{}^a(\partial_b t_{\alpha a}), \tag{65}$$

which provides the freedom to switch what bilinear the derivative operator acts on, when the Lorentz index is summed over. In the special case where $\alpha = \beta$, we can replace four-vectors entirely via

$$j^a(\partial_b j_a) = -m^a(\partial_b m_a) = -n^a(\partial_b n_a) = -k^a(\partial_b k_a) = \sigma(\partial_b\sigma) - \omega(\partial_b\omega), \tag{66}$$

which is just the derivative of the invariant length squared Fierz identity [9]. Note that (66) is consistent with (64), when setting $a = b$ and summing. Applying these identities to the square brackets in (61) and (62) gives, after some manipulation

$$j_c j^e k_d(\partial_b k_e) + m_c m^e n_d(\partial_b n_e)$$
$$= \frac{1}{2}(\sigma^2-\omega^2)[j_c(\partial_b j_d) - k_c(\partial_b k_d) + n_c(\partial_b n_d) + m_c(\partial_b m_d)]. \tag{67}$$

We now write a much simpler form of (58) and (59):

$$\omega[\overline{\psi}\gamma_5\sigma^{ab}(\partial_b\psi) - (\partial_b\overline{\psi})\gamma_5\sigma^{ab}\psi] - \sigma[\overline{\psi}\sigma^{ab}(\partial_b\psi) - (\partial_b\overline{\psi})\sigma^{ab}\psi]$$
$$= \frac{1}{2}\delta^{abcd}[-j_c(\partial_b j_d) + k_c(\partial_b k_d) + n_c(\partial_b n_d) + m_c(\partial_b m_d)], \tag{68}$$

$$\sigma[\overline{\psi}\gamma_5\sigma^{ab}(\partial_b\psi) - (\partial_b\overline{\psi})\gamma_5\sigma^{ab}\psi] - \omega[\overline{\psi}\sigma^{ab}(\partial_b\psi) - (\partial_b\overline{\psi})\sigma^{ab}\psi]$$
$$= \frac{1}{2}\epsilon^{abcd}[-j_c(\partial_b j_d) + k_c(\partial_b k_d) + n_c(\partial_b n_d) + m_c(\partial_b m_d)]. \quad (69)$$

Finally, substituting into (52) and (53), we obtain the gravitational four-vector potentials in terms of *closed-form bilinears* only

$$\Omega^a = \frac{1}{2}(\sigma^2 - \omega^2)^{-1}\{\mathrm{i}[\omega(\partial^a\omega) - \sigma(\partial^a\sigma)] - 2m\omega k^a$$
$$+ \frac{1}{2}\delta^{abcd}[-j_c(\partial_b j_d) + k_c(\partial_b k_d) + n_c(\partial_b n_d) + m_c(\partial_b m_d)]\}, \quad (70)$$
$$\Omega_5^a = \frac{1}{2}(\sigma^2 - \omega^2)^{-1}\{\mathrm{i}[\sigma(\partial^a\omega) - \omega(\partial^a\sigma)] - 2m\sigma k^a$$
$$+ \frac{1}{2}\epsilon^{abcd}[-j_c(\partial_b j_d) + k_c(\partial_b k_d) + n_c(\partial_b n_d) + m_c(\partial_b m_d)]\}. \quad (71)$$

Comparing with the inverted Dirac equation in the electromagnetic case [11]

$$A^a = \frac{1}{2q}(\sigma^2 - \omega^2)^{-1}\{\epsilon^{abcd}[j_c(\partial_b k_d) - k_c(\partial_b j_d)] + m^b(\partial^a n_b) - 2m\sigma j^a\}$$
$$+ \frac{1}{2q}(\sigma^2 - \omega^2)^{-2}\{\delta^{abcd} j_c k_d[\omega(\partial_b\sigma) - \sigma(\partial_b\omega)]$$
$$+ \epsilon^{abcd} j_c k_d[\omega(\partial_b\omega) - \sigma(\partial_b\sigma)]\}, \quad (72)$$

where the totality of the $U(1)$ gauge dependence is represented by the $m^b(\partial^a n_b)$ term, we can see some apparent structural similarities, despite their differences.

5. The Einstein-Cartan-Dirac self-coupled system

5.1. Curvature Field Equations

Consider Einstein's equations coupled to a source term with generally non-vanishing cosmological constant:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \quad (73)$$

In the present case, where the gravitational field couples to the Dirac field, the asymmetric canonical stress-energy tensor on the right hand side is given by [16]

$$T_{\mu\nu} = \frac{\mathrm{i}}{2}[\overline{\psi}\gamma_\mu(\nabla_\nu\psi) - (\nabla_\nu\overline{\psi})\gamma_\mu\psi]. \quad (74)$$

This can be rewritten in terms of Dirac bilinears with the use of Fierz identities [28], which yields

$$T_{\mu\nu} = \frac{1}{2}(\sigma^2 - \omega^2)^{-1}[\mathrm{i}k_\mu(\omega\partial_\nu\sigma - \sigma\partial_\nu\omega) - g^{-1/2}\epsilon_{\mu\sigma\rho\epsilon}(\nabla_\nu j^\sigma)j^\rho k^\epsilon + j_\mu m^\sigma(\nabla_\nu n_\sigma)]. \quad (75)$$

One the other side of the equation, we have the contractions of the curvature tensor, which in terms of the spin connection is given by [21], [29]

$$R^a{}_{b\mu\nu} = \partial_\nu\Gamma_\mu{}^a{}_b - \partial_\mu\Gamma_\nu{}^a{}_b - \Gamma_\mu{}^a{}_e\Gamma_\nu{}^e{}_b + \Gamma_\nu{}^a{}_e\Gamma_\mu{}^e{}_b. \quad (76)$$

It is important to note that the curvature tensor we use is not the Riemannian one from standard general relativity, due to the presence of a non-vanishing torsion field. The non-Riemannian component of $R^a{}_{b\mu\nu}$ vanishes in the limit where the torsion vanishes. An expression in terms of locally orthonormal components is obtained, as usual, via contraction with the vierbein

$$\begin{aligned} R^{ab}{}_{cd} &\equiv e^\mu{}_c e^\nu{}_d R^{ab}{}_{\mu\nu} \\ &= [e^\mu{}_c(\partial_d e_\mu{}^e) - e^\mu{}_d(\partial_c e_\mu{}^e)]\Gamma_e{}^{ab} + \partial_d\Gamma_c{}^{ab} - \partial_c\Gamma_d{}^{ab} - \Gamma_c{}^a{}_e\Gamma_d{}^{eb} + \Gamma_d{}^a{}_e\Gamma_c{}^{eb}. \end{aligned} \tag{77}$$

Switching the derivatives on the vierbein terms (which reverses the sign), we can write the curvature tensor as

$$R^{ab}{}_{cd} = \Theta^e{}_{cd}\Gamma_e{}^{ab} + \partial_d\Gamma_c{}^{ab} - \partial_c\Gamma_d{}^{ab} - \Gamma_c{}^a{}_e\Gamma_d{}^{eb} + \Gamma_d{}^a{}_e\Gamma_c{}^{eb}, \tag{78}$$

where we define the *objects of anholonomity* as

$$\Theta_{abc} \equiv e_{\mu a}(\partial_b e^\mu{}_c - \partial_c e^\mu{}_b), \tag{79}$$

which are representative of the non-commutativity of the tetrad basis [17]. Contracting b and d in the curvature tensor yields the Ricci tensor

$$R^a{}_b = \Theta^c{}_{bd}\Gamma_c{}^{ad} + \partial_c\Gamma_b{}^{ac} - \partial_b\Gamma_c{}^{ac} - \Gamma_b{}^a{}_c\Gamma_d{}^{cd} + \Gamma_d{}^a{}_c\Gamma_b{}^{cd}, \tag{80}$$

with the final contraction yielding the Ricci scalar

$$R = \Theta_{abc}\Gamma^{abc} + 2\partial_a\Gamma_b{}^{ba} - \Gamma_a{}^a{}_b\Gamma_c{}^{bc} + \Gamma_{abc}\Gamma^{bca}. \tag{81}$$

5.2. Torsion Field Equations

The torsion tensor is defined as the degree to which the affine connection fails to be symmetric:

$$\Upsilon_{\mu\nu}{}^\lambda = \Gamma_{\mu\nu}{}^\lambda - \Gamma_{\nu\mu}{}^\lambda. \tag{82}$$

A particle field with intrinsic quantum spin will act as the source of a non-vanishing torsion field, in an analogous manner to stress-energy acting as the source of curvature [15]. The torsion field equation is given by

$$\Upsilon_{\mu\nu}{}^\gamma + \delta^\gamma_\mu\Upsilon_{\nu\sigma}{}^\sigma - \delta^\gamma_\nu\Upsilon_{\mu\sigma}{}^\sigma = 8\pi G\Sigma_{\mu\nu}{}^\gamma. \tag{83}$$

Together, the curvature and torsion gravitational field equations, (73) and (83), comprise the Einstein-Cartan(-Sciama-Kibble) equations.

In terms of the spinor field, the canonical spin momentum tensor in a locally orthonormal frame is

$$\Sigma^{abc} = \frac{\mathrm{i}}{4}\overline{\psi}\gamma^{[a}\gamma^b\gamma^{c]}\psi. \tag{84}$$

Given that

$$\gamma^{[a}\gamma^b\gamma^{c]} = \frac{1}{6}(\gamma^a\gamma^b\gamma^c - \gamma^a\gamma^c\gamma^b + \gamma^b\gamma^c\gamma^a - \gamma^b\gamma^a\gamma^c + \gamma^c\gamma^a\gamma^b - \gamma^c\gamma^b\gamma^a), \tag{85}$$

we can apply the Dirac identities

$$\gamma^a\gamma^b = 2\eta^{ab} - \gamma^b\gamma^a \tag{86a}$$

$$\gamma^a\sigma^{bc} = \mathrm{i}\eta^{ab}\gamma^c - \mathrm{i}\eta^{ac}\gamma^b + \epsilon^{abcd}\gamma_5\gamma_d \tag{86b}$$

to obtain

$$\gamma^{[a}\gamma^b\gamma^{c]} = -\mathrm{i}\epsilon^{abcd}\gamma_5\gamma_d. \tag{87}$$

Substituting into (84), we find

$$\Sigma^{abc} = \frac{1}{4}\epsilon^{abcd}k_d, \tag{88}$$

the spin angular momentum tensor of the Dirac field is proportional to the rank-3 dual of the axial vector bilinear. With regards to the left-hand side of (83), using the connection transformation rule (12), we can write the torsion in terms of the object of anholonomity and connection

$$\Upsilon_{abc} \equiv \Upsilon_{\mu\nu}{}^{\lambda}e^{\mu}{}_a e^{\nu}{}_b e_{\lambda c} = \Theta_{cba} - \Gamma_{abc} + \Gamma_{bac}. \tag{89}$$

Alternatively, taking an appropriate cyclic combination of the torsion, the connection can be written as [30], [29]

$$\Gamma_{abc} = K_{abc} + \frac{1}{2}(\Theta_{abc} - \Theta_{bca} - \Theta_{cab}), \tag{90}$$

where we define the *contorsion* tensor to be

$$K_{abc} := \frac{1}{2}(-\Upsilon_{abc} + \Upsilon_{bca} - \Upsilon_{cab}). \tag{91}$$

5.3. Constraints Arising From Torsion

We shall now demonstrate how the torsion field equation can be used to obtain a further, very useful set of constraints on the Einstein-Cartan-Dirac system. For convenience, we shall consider the torsion field equation in a local frame

$$\Upsilon_{abc} + \eta_{ac}\Upsilon_b{}^d{}_d - \eta_{bc}\Upsilon_a{}^d{}_d = 8\pi G\Sigma_{abc}. \tag{92}$$

Now, substituting the irreducible decomposition of the connection (30) into the torsion (89), we obtain

$$\Upsilon_{abc} = -\Theta_{cab} + {}^{(3)}\Gamma_{cab} - \frac{2}{3}\delta_{cdab}\Omega^d + \frac{4}{3}\epsilon_{cdab}\Omega_5^d, \tag{93}$$

where we have used the cyclic identities

$$\delta_{adbc} - \delta_{bdac} = -\delta_{cdab}, \tag{94}$$

$${}^{(3)}\Gamma_{abc} - {}^{(3)}\Gamma_{bac} = -{}^{(3)}\Gamma_{cab}. \tag{95}$$

Taking the trace of (93) in the last two indices, the ${}^{(3)}\Gamma$ and Ω_5 terms vanish, and we find

$$\Upsilon_a{}^b{}_b = \Theta_b{}^b{}_a + 2\mathrm{i}\Omega_a, \tag{96}$$

where we have used the antisymmetry of Θ_{abc} in bc. Substituting (93), (96), and (88) into (92), then gathering terms and rearranging, we obtain an explicit expression for the remaining component of the connection

$$ {}^{(3)}\Gamma_{abc} = \Theta_{abc} + \mathrm{i}\delta_{adbc}\Theta_e{}^{ed} - \frac{4}{3}\delta_{adbc}\Omega^d - \frac{4}{3}\epsilon_{abcd}\Omega_5^d + 2\pi G\epsilon_{abcd}k^d. \tag{97} $$

Taking the trace of (97), the left-hand side and Levi-Civita terms vanish, and we obtain the constraint on the gravitational vector potential

$$ \Omega_a = \frac{\mathrm{i}}{2}\Theta_b{}^b{}_a. \tag{98} $$

Similarly, when we fully contract both sides of (97) with the Levi-Civita tensor, the left-hand side and δ-dependent terms vanish. Using the Levi-Civita contraction identity

$$ \epsilon_{abcd}\epsilon^{abcf} = -6\delta_d{}^f, \tag{99} $$

where the factors of $\sqrt{|g|}$ cancel out, we obtain the constraint on the dual gravitational potential

$$ \Omega_5^d = -\frac{1}{8}\Theta_{abc}\epsilon^{abcd} + \frac{3\pi}{2}Gk^d. \tag{100} $$

Substituting our constraints (98) and (100) back into (97), we obtain the expression

$$ {}^{(3)}\Gamma_{abc} = \frac{1}{3}(2\Theta_{abc} - \Theta_{bca} - \Theta_{cab}) + \frac{\mathrm{i}}{3}\delta_{adbc}\Theta_e{}^{ed}. \tag{101} $$

This expression can be interpreted as that due to the constraints imposed by the Cartan torsion equation (92), the traceless mixed symmetry irreducible component of the connection is equal to the traceless mixed symmetry irreducible component of the object of anholonomity. Substituting the constraints (98), (100), and (101) into (93), we obtain the simple form of the torsion

$$ \Upsilon_{abc} = 8\pi G\Sigma_{abc}, \tag{102} $$

which is obviously a solution of (92) due to the vanishing trace of the fully antisymmetric spin tensor. Substituting the same constraints into the connection (30) we obtain

$$ \Gamma_{abc} = -4\pi G\Sigma_{abc} + \frac{1}{2}(\Theta_{abc} - \Theta_{bca} - \Theta_{cab}), \tag{103} $$

which is consistent with (90), and the contorsion solution corresponding to (102):

$$ K_{abc} = -4\pi G\Sigma_{abc}. \tag{104} $$

5.4. The Dirac Bilinear Local Frame

In section 4, we used the fact that there is a local orthonormal tetrad frame corresponding to a family of observers comoving with the Dirac matter. Using the Fierz identities for the four four-vector quantities§ derived from the Dirac algebra [9], [31], [7], [10]

$$ j_\mu j^\mu = -m_\mu m^\mu = -n_\mu n^\mu = -k_\mu k^\mu = \sigma^2 - \omega^2, \tag{105} $$

$$ j_\mu m^\mu = j_\mu n^\mu = j_\mu k^\mu = m_\mu n^\mu = m_\mu k^\mu = n_\mu k^\mu = 0, \tag{106} $$

§ Subjecting the spinors to a complex phase transformation, causes the m-n plane to rotate by an angle corresponding to double the phase parameter, whereas the j and k vectors are left invariant.

where these bilinears are defined in terms of the spinors in (36a)-(38c), a local tetrad frame

$$(\boldsymbol{t}_0, \boldsymbol{t}_1, \boldsymbol{t}_2, \boldsymbol{t}_3) = (\sigma^2 - \omega^2)^{-1/2}(\boldsymbol{j}, \boldsymbol{m}, \boldsymbol{n}, \boldsymbol{k}) \\ = (\sigma^2 - \omega^2)^{-1/2}(j^\mu, m^\mu, n^\mu, k^\mu)\partial_\mu, \tag{107}$$

can be constructed, with the time-like direction given by $\boldsymbol{j}$, and the three space-like directions given by $\boldsymbol{m}$, $\boldsymbol{n}$, and $\boldsymbol{k}$, with normalising factor $(\sigma^2 - \omega^2)^{-1/2}$ equal to the reciprocal of the invariant length of the four-vectors via (105). This bilinear tetrad is one of infinitely many local orthonormal frames, related to one another by a local Lorentz transformation. It is pertinent to ask whether we are able to describe the components of an arbitrary vierbein field in the coordinate frame ${e_a}^\mu$ in terms of the bilinears which appear in ${t_a}^\mu$. Consider an arbitrary four-vector field $\boldsymbol{V}$ in terms of the coordinate frame (c), and the bilinear (b) and generic (g) tetrad local frames

$$\boldsymbol{V} = V^{(c)\mu}\partial_\mu = V^{(b)i}\boldsymbol{t}_i = V^{(g)a}\boldsymbol{e}_a. \tag{108}$$

Taking all right-hand parts of the equation with respect to the coordinate frame gives the relationship between the various components

$$V^{(c)\mu} = {t_i}^\mu V^{(b)i} = {e_a}^\mu V^{(g)a}. \tag{109}$$

Contracting (109) with the inverse of the bilinear tetrad gives

$$V^{(b)i} = {t^i}_\mu {e_a}^\mu V^{(g)a} = {t^i}_a V^{(g)a}, \tag{110}$$

Substituting (110) into (109) yields an expression for the generic vierbein frame in terms of the Dirac bilinear frame

$${e_a}^\mu = {t_i}^\mu {t^i}_a = (\sigma^2 - \omega^2)^{-1}(j_a j^\mu - m_a m^\mu - n_a n^\mu - k_a k^\mu). \tag{111}$$

This expression for the generic vierbein field in terms of the Dirac bilinears provides us with a tool for calculating *internal* solutions of the Einstein-Cartan-Dirac equations, in regions where the Dirac field is non-vanishing; vacuum solutions must be matched on the matter boundary. The expression (111) simplifies the self-coupled equations summarized below in section 6 by reducing the number of fields we need to solve for, to the number of independent parameters in the bilinears. On the other hand, due to the length of the right-hand side of (111) the equations may appear far more untidy. However, this will be offset by the application of Fierz identities between contracted bilinears. An explicit algebraic reduction is deferred to future publications.

6. Summary and conclusions

For the sake of clarity, we shall collate our main results. We have Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{112}$$

where on the right-hand side, we have the Dirac matter stress-energy tensor

$$T_{\mu\nu} = \frac{1}{2}(\sigma^2 - \omega^2)^{-1}[\mathrm{i}k_\mu(\omega\partial_\nu\sigma - \sigma\partial_\nu\omega) - g^{-1/2}\epsilon_{\mu\sigma\rho\epsilon}(\nabla_\nu j^\sigma)j^\rho k^\epsilon + j_\mu m^\sigma(\nabla_\nu n_\sigma)], \tag{113}$$

and on the left, we have the Ricci tensor and scalar, which in the local frame are respectively

$$R^a{}_b = \Theta^c{}_{bd}\Gamma_c{}^{ad} + \partial_c\Gamma_b{}^{ac} - \partial_b\Gamma_c{}^{ac} - \Gamma_b{}^a{}_c\Gamma_d{}^{cd} + \Gamma_d{}^a{}_c\Gamma_b{}^{cd}, \tag{114}$$

$$R = \Theta_{abc}\Gamma^{abc} + 2\partial_a\Gamma_b{}^{ba} - \Gamma_a{}^a{}_b\Gamma_c{}^{bc} + \Gamma_{abc}\Gamma^{bca}. \tag{115}$$

Note that our curvature terms implicitly contain a non-zero torsion component. The covariant derivatives in the stress-energy tensor contain the connection with world indices, which due to its inhomogeneous transformation law, can be written in terms of the local frame as

$$\Gamma_{\mu\nu}{}^\lambda = e^a{}_\mu e_{b\nu} e_c{}^\lambda \Gamma_a{}^{bc} + e^a{}_\mu e_b{}^\lambda \partial_a e^b{}_\nu. \tag{116}$$

Reducing the connection into three irreducible terms, and taking account of the torsion equation, the connection in the local frame can be written as

$$\Gamma_{abc} = -\frac{2}{3}\delta_{adbc}\Omega^d - \frac{2}{3}\epsilon_{abcd}\Omega_5^d + \frac{1}{3}(2\Theta_{abc} - \Theta_{bca} - \Theta_{cab}) + \frac{\mathrm{i}}{3}\delta_{adbc}\Theta_e{}^{ed}, \tag{117}$$

where the first two terms can be written in terms of Dirac bilinears, via the gravitational vector potentials obtained by inverting the Dirac equation:

$$\begin{aligned}\Omega^a = &\frac{1}{2}(\sigma^2 - \omega^2)^{-1}\{\mathrm{i}[\omega(\partial^a\omega) - \sigma(\partial^a\sigma)] - 2m\omega k^a \\ &+ \frac{1}{2}\delta^{abcd}[-j_c(\partial_b j_d) + k_c(\partial_b k_d) + n_c(\partial_b n_d) + m_c(\partial_b m_d)]\},\end{aligned} \tag{118}$$

$$\begin{aligned}\Omega_5^a = &\frac{1}{2}(\sigma^2 - \omega^2)^{-1}\{\mathrm{i}[\sigma(\partial^a\omega) - \omega(\partial^a\sigma)] - 2m\sigma k^a \\ &+ \frac{1}{2}\epsilon^{abcd}[-j_c(\partial_b j_d) + k_c(\partial_b k_d) + n_c(\partial_b n_d) + m_c(\partial_b m_d)]\}.\end{aligned} \tag{119}$$

The object of anholonomity is given in terms of the vierbein as

$$\Theta_{abc} \equiv e_{\mu a}(\partial_b e^\mu{}_c - \partial_c e^\mu{}_b). \tag{120}$$

Using the fact that, due to the existence of a local orthonormal frame carried by observers co-moving with the Dirac field, the vierbein field can be written as

$$e_a{}^\mu = (\sigma^2 - \omega^2)^{-1}(j_a j^\mu - m_a m^\mu - n_a n^\mu - k_a k^\mu), \tag{121}$$

implying that the object of anholonomity can also be described using only bilinears; via (117) the *connection* in the local frame can also be described using only bilinears. Furthermore, the torsion field equation provides us with the constraints

$$\Omega_a = \frac{\mathrm{i}}{2}\Theta_b{}^b{}_a, \tag{122}$$

$$\Omega_5^d = -\frac{1}{8}\Theta_{abc}\epsilon^{abcd} + \frac{3\pi}{2}Gk^d, \tag{123}$$

$$\Gamma_{abc} = -\pi G\epsilon_{abcd}k^d + \frac{1}{2}(\Theta_{abc} - \Theta_{bca} - \Theta_{cab}). \tag{124}$$

Taken together, the equations (112)-(124) describe the gravitationally self-interacting Einstein-Cartan-Dirac equations, in terms of the Lorentz covariant observables of the Dirac field: the Dirac bilinears. We believe the inverted forms of the Dirac equation (118) and

(119), the Fierz identities (68) and (69) that lead to their description in terms of Dirac bilinears as opposed to spinors, and their application to the Einstein-Cartan-Dirac system, to be new results.

In the electromagnetic case of the self-coupled Maxwell-Dirac equations, we showed that this system is able to be reduced in the presence of global spacetime symmetries corresponding to subgroups of the Poincaré group, and we gave four specific examples [11]. The approach we used was an infinitesimal method, which involved using the Lie generators of a particular Poincaré subalgebra, provided by Patera, Winternitz & Zassenhaus [12], to calculate joint invariant scalar and vector fields, which were then applied to the physical equations to obtain new exact and numerical solutions [32]. Due to the similar complexity of the Einstein-Cartan-Dirac equations, global symmetry reduction using the same techniques is one way in which solutions to this system can be pursued.

Another avenue of study which the results of this paper highlight is that of the extended Fierz algebra. The derivation of the Fierz identities needed to manipulate expressions involving Dirac bilinears, for the case where the spinors carry no tensor indices is straightforward (see (54)-(57)). However, the Dirac equation and related expressions of course involve partial derivatives of spinors, so that new classes of "higher rank" Fierz identities must be obtained. Equations (68) and (69) are two examples of a much broader set of such relations.

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Appendix A. Glossary of symbols

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Table A1. Glossary of symbols used in this article.

Symbol	Description
$\mu,\nu,...=0,1,2,3$	Global coordinate frame indices
$a,b,...=0,1,2,3$	Local orthonormal frame indices
$g_{\mu\nu}$	Global coordinate frame metric
$\eta_{ab}=diag(1,-1,-1,-1)$	Local orthonormal frame metric
${e_a}^{\mu}$	Vierbein coordinate components
γ^a	Gamma matrices
$\gamma^5=\gamma_5=\mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$	Gamma-5 matrix
$\sigma_{ab}=\frac{\mathrm{i}}{2}[\gamma_a,\gamma_b]$	Rank-2 gamma matrix commutator
$\Gamma_{R=1,...,16}=\{I,\gamma^a,\sigma^{ab},\gamma_5\gamma^a,\gamma_5\}$	Dirac-Clifford algebra basis
ϵ^{abcd}	Levi-Civita symbol
$\delta^{abcd}:=\mathrm{i}(\eta^{ac}\eta^{bd}-\eta^{ad}\eta^{bc})$	Sylvester tensor
$\Gamma_\mu=\frac{1}{2}{\Gamma_\mu}^{ab}S_{ab}$	Spinor connection
$S_{ab}=-\frac{\mathrm{i}}{2}\sigma_{ab}$	Lorentz generators in Dirac spinor rep.
${\Gamma_{\mu\nu}}^{\lambda}$	Riemann-Cartan connection
ψ,χ	Dirac spinors
$\overline{\psi}:=\psi^\dagger\gamma^0$	Dirac conjugate spinor
$\psi^{\mathrm{c}}:=C\overline{\psi}^{\mathrm{T}}=\mathrm{i}\gamma^2\gamma^0\overline{\psi}^{\mathrm{T}}$	Dirac charge conjugate spinor
${R^{\mu}}_{\nu\sigma\lambda}$	Riemann-Cartan curvature tensor
$R_{\mu\nu}$	Ricci tensor
R	Ricci scalar
Λ	Cosmological constant
$\delta^\mu_\nu:=diag(1,1,1,1)$	Kronecker delta
${\Upsilon_{\mu\nu}}^{\lambda}:=\Gamma^\lambda_{\mu\nu}-\Gamma^\lambda_{\nu\mu}$	Torsion tensor
$T^{\mu\nu}$	Stress-energy tensor
$\Sigma^{\mu\nu\lambda}$	Spin angular momentum density tensor
$\Omega_d:=\frac{1}{4}\delta_{adbc}\omega^{abc}$	Gravitational vector potential
$\Omega_{5d}:=\frac{1}{4}\epsilon_{abcd}\omega^{abc}$	Gravitational axial vector potential
A_a	Electromagnetic vector potential
A_{5a}	Abelian axial vector potential
$\sigma:=\overline{\psi}\psi$	Dirac bilinear scalar field
$\omega:=\overline{\psi}\gamma_5\psi$	Dirac bilinear pseudoscalar field
$j^a:=\overline{\psi}\gamma^a\psi$	Dirac bilinear vector field
$k^a:=\overline{\psi}\gamma_5\gamma^a\psi$	Dirac bilinear axial vector field
$s^{ab}:=\overline{\psi}\sigma^{ab}\psi$	Dirac bilinear rank-2 tensor field
${}^*s^{ab}:=\overline{\psi}\gamma_5\sigma^{ab}\psi$	Dirac bilinear dual rank-2 tensor field
$m^a:=\mathrm{Re}\{\overline{\psi}^{\mathrm{c}}\gamma^a\psi\}$	Complex Dirac bilinear, real part
$n^a:=\mathrm{Im}\{\overline{\psi}^{\mathrm{c}}\gamma^a\psi\}$	Complex Dirac bilinear, imaginary part
${t_a}^{\mu}:=(\sigma^2-\omega^2)^{-1/2}[j^\mu,m^\mu,n^\mu,k^\mu]$	Dirac tetrad frame, coord. components
$\Theta_{abc}:=e_{a\mu}(\partial_b{e_c}^{\mu}-\partial_c{e_b}^{\mu})$	Object of anholonomity
$K_{abc}:=\frac{1}{2}(-\Upsilon_{abc}+\Upsilon_{bca}-\Upsilon_{cab})$	Contorsion tensor

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