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THE KURATOWSKI CLOSURE-COMPLEMENT THEOREM

B.J. Gardner and M. Jackson

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1. Introduction

The Kuratowski Closure-Complement Theorem 1.1. [**29**] *If $(X, \mathfrak{T})$ is a topological space and $A \subseteq X$ then at most 14 sets can be obtained from A by taking closures and complements. Furthermore there is a space in which this bound is attained.*

This remarkable little theorem and related phenomena have been the concern of many authors. Apart from the mysterious appearance of the number 14, the attraction of this theorem is that it is simple to state and can be examined and proved using concepts available after any first encounter with topology. The goal of this article is both to provide an original investigation into variations of the theorem and its relation to properties of spaces and to survey the existing literature in this direction.

The most convenient approach to investigating the Kuratowski closure-complement theorem is via operator notation. If $(X, \mathfrak{T})$ is a topological space we will consider the complement operator a acting on the set of subsets of X defined by $a(A) = X \backslash A$ and the closure operator b defined by $b(A) = \bar{A}$. We will also use the symbol i to denote the interior of a set: $i(A) = a(b(a(A)))$. We prefer the neutral symbols a and b for complement and closure since expressions built from iterated dashes and bars look awkward, while the notation $C(A)$ could mean either complement or closure. The investigation of topology from the point of view of topological closure operators on a set was begun by Kuratowski.

Given a set $\mathcal{O} := \{o_i : i \in I\}$ of operators on a set S we may construct some possibly new operators by composing the members of $\mathcal{O}$. As composition is associative, the set of distinct operators that arises forms a monoid under composition, with identity element id. Here two operators o_1 and o_2 are distinct if there is an $s \in S$ such that $o_1(s) \neq o_2(s)$. We will refer to the monoid of operators generated by closure and complement on a topological space $(X, \mathfrak{T})$ as the *Kuratowski monoid* of $(X, \mathfrak{T})$ and to the operators themselves as *Kuratowski operators*. Since we will almost always be considering operators on topological spaces it will be useful to refer to an operator that always produces a closed set (or open set) as a *closed operator* (or *open operator* respectively).

There is a natural partial order on a monoid $\{o_i : i \in I\}$ of operators on the subsets of a set X defined by $o_1 \leq o_2$ if for every $A \subseteq X$ we have $o_1(A) \subseteq o_2(A)$. For example, we have $\mathsf{id} \leq b$. In general we say that an operator o is *isotonic* if $X \subseteq Y \Rightarrow o(X) \subseteq o(Y)$ and is *idempotent* if $o^2 = o$. With these ideas in hand it is now quite easy to give a proof of the Kuratowski theorem (see also

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[**29, 21, 30, 15, 8**]). We present this proof for completeness.

Proof of the Kuratowski Closure-Complement Theorem. First note that the operator a satisfies $a^2 = \mathsf{id}$ (where id is the identity operator), while b is idempotent: $b^2 = b$. This immediately shows that every operator in the operator monoid generated by a and b is either the identity operator id or is equal to one of the form $abab\dots ba$, $baba\dots ba$, $abab\dots ab$, or $baba\dots ab$. Our first goal is to show that the operators bab and $bababab$ are identical. This will give the upper bound of 14 since the only remaining operators are

$$\mathsf{id}, a, b, ab, ba, aba, bab, abab, baba, ababa, babab, ababab, bababa, abababa.$$

Note that $bab \geq ababab$ since $ababab(A)$ is the interior of $bab(A)$. Because b is isotonic it then follows that $bab = bbab \geq bababab$. On the other hand, $abab \leq b$ (since $abab(A)$ is the interior of $b(A)$) and so $babab \leq bb = b$. But then $ababab \geq ab$ and $bababab \geq bab$. Combining these two inequalities gives $bab = bababab$ as required.

To complete the proof it suffices to find a topological space with a subset for which each of the 14 possible operators produces a different set. Finding a subset of the reals with this property is an exercise is some textbooks [**27**, p. 57]. As shown in [**44**, Example 32, part 9], one example is the set

$$\{1/n \mid \ n \in \mathbb{N}\} \cup (2,3) \cup (3,4) \cup \{4\frac{1}{2}\} \cup [5,6] \cup \{x \mid \ x \text{ is rational and } 7 \leq x < 8\}$$

(but in fact only the 2$^{\text{nd}}$, 3$^{\text{rd}}$, 4$^{\text{th}}$ and 6$^{\text{th}}$ components are necessary). This completes the proof. □

An equivalent formulation of the Kuratowski closure-complement theorem is the statement that at most 7 distinct sets can be obtained from a subset of a topological space by applying closures and interiors. Certainly the earlier formulation implies this. However if we have 7 distinct sets obtained from some subset A using closure and interior then it is easily verified that the complement of each of these sets gives a new set. In fact we may (as in [**29**]) divide the Kuratowski operators into two subsets; those that involve an even number of complements (the *even* Kuratowski operators) and those that involve an odd number (the *odd* Kuratowski operators). The natural order on the even Kuratowski operators is given in Figure 1.1. All odd Kuratowski operators can be obtained by applying a complement to one side of an even operator (obviously the order is reversed by complementation).

Definition 1.2. *Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$.*

(i) $k(A)$ *(the* k-number *of* A*) denotes the number of distinct sets obtainable from* A *by taking closures and complementation. A set with* k*-number* n *will also be called an* n-set.
(ii) $k((X, \mathcal{T}))$ *(the* k-number *of* $(X, \mathcal{T})$*) denotes* $\max\{k(A) : A \subseteq X\}$.
(iii) $K((X, \mathcal{T}))$ *(the* K-number of $(X, \mathcal{T})$*) denotes the number of distinct Kuratowski operators on* $(X, \mathcal{T})$*; that is, the order of the Kuratowski monoid of* $(X, \mathcal{T})$.

It can be seen that the Kuratowski result has two facets: it shows that the k-number of any set in a topological space is at most 14 and that the K-number of any

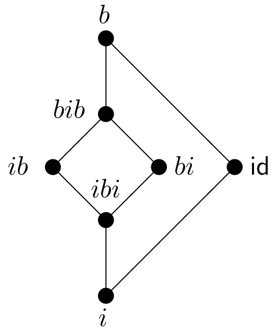


FIGURE 1.1. The seven different operators using b and i.

topological space is at most 14. Our proof of the Kuratowski closure-complement theorem first showed that $K((X,\mathfrak{T})) \leq 14$ and then that the k-number of the reals with the usual topology is actually 14. Naturally, $K((X,\mathfrak{T})) \geq k((X,\mathfrak{T}))$. The following definitions will also be useful.

Definition 1.3. *A space with K-number* 14 *will be called a* Kuratowski space. *A space with k-number equal to its K-number will be called a* full space.

Section 2 deals with characterising spaces by their Kuratowski monoid, their K-number and their k-number; while these topics have been commonly investigated in the literature, we make some original contributions too. Section 3 deals with the stability of the various Kuratowski phenomena under topological constructions; this section is mostly new. We investigate how the Kuratowski monoids and k-numbers of spaces change under taking products, subspaces, and continuous images. For example, we find the initially surprising result that every topological space is a subspace of one whose K-number is 6. Finally, in Section 4 we look at variants of Kuratowski's original result. This has also been a popular topic in the literature and we survey many results of both topological and non-topological character.

2. Characterising Spaces and Sets Using Kuratowski-type Phenomena

Classifying spaces and sets according to the behaviour of Kuratowski operators has been a common theme in the literature. We consider three different approaches.

2.1. Classifying spaces by their Kuratowski monoid. Classifying spaces by their Kuratowski monoid appears to be the most natural of the possible ways of making use of the Kuratowski theorem. Despite this, the approach appears to have only been considered by Chagrov in [**10**], a paper that is not readily available, and not available at all in English. Furthermore, the proof that the various Kuratowski monoids are the only possible Kuratowski monoids, which is not quite trivial, is not given in [**10**]. In fact for non-topological closure operators, two possible extra cases emerge. Because of these facts we give a full proof of the Chagrov result.

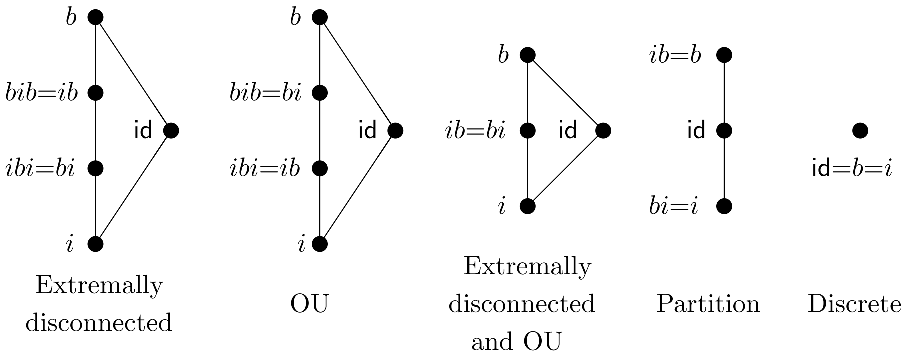


FIGURE 2.1. Orderings on 5 possible Kuratowski monoids.

Theorem 2.1. (*Chagrov,* [**10**]) *The Kuratowski monoid of a non-empty topological space corresponds to either the diagram in Figure 1.1 or one of the 5 diagrams in Figure 2.1. Thus the possible K-numbers of a topological space are* $14, 10, 8, 6, 2$.

Proof. Note that if $o_1 \leq o_2 \leq o_3$ are even Kuratowski operators and $o_1 = o_3$ on some space $(X, \mathfrak{T})$ then we also have $o_1 = o_2 = o_3$ on $(X, \mathfrak{T})$. This means that all collapses of the diagram in Figure 1.1 are created by combinations of collapses between adjacent members in the diagram or between incomparable members of the diagram (that is, pairs of members that are not related by $\leq$ or $\geq$).

Note that the map $o \mapsto aoa$ is (monoid) automorphism of the Kuratowski monoid that interchanges i and b and reverses order, and thus inverts the diagram in Figure 1.1. This ensures that for every possible collapse, the diagram that replaces that in Figure 1.1 will have similar symmetry.

The number of cases to consider is further reduced by the fact that any collapse for which id is equal to a distinct Kuratowski operator o is equivalent to collapsing all operators onto id; this is because if o is an open (closed) operator then $\mathsf{id} = o$ implies that every set is open. Obviously, discrete spaces correspond to the last diagram in Figure 2.1. The only remaining collapses we need consider are $ibi = bi$, $ibi = ib$, $bi = ib$, $i = ibi$ and combinations of these. When reading the remainder of this proof the reader may be interested in consulting Theorem 2.3 below, where we give structural characterisations of these spaces.

Case 1. $ibi = bi$.
We have $aibia = abia$, that is, $aababab aa = ababaa$, which gives $babab = abab$, or $bib = ib$. (This illustrates the symmetry alluded to above.) Since $ibi \leq bib$ this gives the first diagram in Figure 2.1.

Case 2. $ibi = ib$.
By symmetry we have have $bib = bi$. Since $ibi \leq bib$ this gives the second diagram in Figure 2.1. Chagrov comments that these spaces have a connection with the 'weak excluded middle' in logic.

Case 3. $ib = bi$.
This implies that $ib = iib = ibi = bii = bi = bbi = bib$ giving the third diagram in Figure 2.1. Clearly this is also equivalent to case 1 and 2 holding simultaneously.

Case 4. $ibi = i$.
The fact that this case corresponds to the fourth diagram in Figure 2.1 requires a

more detailed argument. Symmetry again gives us $bib = b$, however it also turns out that $ib = b$ and $bi = i$ which follow from the additive properties of topological closure and which are not expressed by the operator monoid structure. It will suffice to show that every closed set is open.

For each $x \in X$, let S_x denote the set $i(X \backslash \{x\})$. If G is an open set, then either $G \subseteq S_x$ or $x \in G$; therefore $S_x \cup \{x\}$ is dense in X. Now $S_x = i(S_x) = ibi(S_x)$ by assumption and therefore $b(S_x) = bi(S_x) \neq X$. These facts imply that $x \notin b(S_x)$.

Now consider the open set $G_x := X \backslash b(S_x)$ which does contain x. Let $y \in G_x$ and H be an open set containing y. Since $H \not\subseteq X \backslash G_x = b(S_x)$, we have that $H \not\subseteq ib(S_x) = ibi(S_x) = i(S_x) = S_x$. Since S is the union of all open sets not containing x, it follows that $x \in H$. That is, every open set containing y contains x, or equivalently $y \in b(\{x\})$. Because y was an arbitrary element of G_x it follows that $b(G_x) \subseteq b(\{x\}) \subseteq b(G_x)$.

Now to show that every closed set is open. Let K be a closed set containing an element x. So $b(\{x\}) \subseteq K$. But then $x \in G_x \subseteq b(G_x) = b(\{x\}) \subseteq K$ showing that $K = \cup_{x \in K} G_x$ is a union of open sets and therefore open.

Since case 4 turns out to be a further collapse of case 3, it remains to show that case 2 and case 4 holding simultaneously does not imply any new kind of space. In case 2 we have $i \leq ib \leq bi$ and $i \leq \mathsf{id}$ while in case 4 we have $i = bi$ and $\mathsf{id} \leq b = ib$. Hence if both case 2 and case 4 hold then we have $i = b = \mathsf{id}$, that is, a discrete space.

To complete the proof we need to show that that each of the 6 possibilities does arise. Chagrov gives examples of small finite topologies; we instead aim for topologies that are either well known or that arise 'naturally'. Obviously for K-number 14, we may take the reals with the usual topology.

For an example of a space satisfying $ibi = bi$ but not $ibi = ib$ or $ibi = i$, one may take the topology of all cofinite subsets of an infinite set (with the empty set). As we explain below, spaces satisfying $ibi = bi$ are called extremally disconnected spaces; some well known extremally disconnected Hausdorff spaces can be found in [**44**] for example.

For an example of a space satisfying $ibi = ib$ (but not $ibi = bi$), let X be a set with $|X| > 2$ and let $x \in X$ be fixed. We define $\mathcal{T} := \{S \subseteq X : x \notin S \text{ or } S = X\}$.

For an example of a space satisfying $ibi = i$, one may consider an equivalence relation θ on a set X and then take the empty set and the equivalence classes of θ as a basis. $\square$

The example of a space satisfying $ibi = bi$ given in [**10**] is the set $\{a, b, c, d, e\}$ with the basis $\{\varnothing, \{a\}, \{b\}, \{a, c\}, \{b, d\}, \{a, c, e\}, \{b, d, e\}\}$. In fact this family does not form a basis since $\{a, c, e\} \cap \{b, d, e\} = \{e\}$. Taking the family as a subbasis it can be verified that the resulting space also satisfies $ibi = ib$ and therefore has a K-number of 8 (not 10 as claimed).

The example of an space satisfying $ibi = ib$ in [**10**] is also incorrect. Here we have the set $\{a, b, c, d, e\}$ with basis $\{\varnothing, \{a, e\}, \{a, b, e\}, \{a, c, e\}, \{a, b, c, d, e\}\}$. Every open set is dense in this space and therefore the space satisfies $ibi = bi$ and in fact it does not satisfy $ibi = ib$.

Case 4 in the proof of Theorem 2.1 involved some topological arguments. If one restricts to the purely algebraic properties of the Kuratowski monoid, two further collapses of the diagram in Figure 1.1 are possible; these are given in Figure 2.2.

For an example of an operator on a set with the first operator monoid consider the

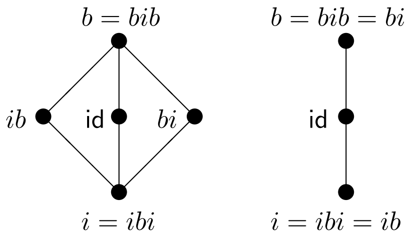


FIGURE 2.2. Two nearly possible Kuratowski monoids.

two element group $\mathbb{Z}_2$ and for $A \subseteq \{0,1\}$ let $b(A)$ denote the subgroup generated by A. Note that the operator aba associates each set with the complement of a subgroup and therefore does not contain 0 (which is contained in every subgroup). Hence we cannot have $bi = ibi$ since bi gives a subgroup while ibi does not. The law $ibi = i$ is routinely verified.

For an example of the second kind, consider a set X with $b(A) = X$ for all $A \subseteq X$.

These two new possibilities may not be of interest from a topological perspective, but might perhaps be useful in the classification of non-topological closures. We discuss several natural candidates in the final section of this paper.

The possible topological collapses are all given by simple equalities between even Kuratowski operators. We now give slightly more structural descriptions. First recall some definitions.

Definition 2.2. (i) *A space is* extremally disconnected *if the closure of any open set is open* [**46**].

(ii) *A space is* resolvable *if it contains a dense set with empty interior* [**23**].

(iii) *A space is an* open unresolvable space *if no open subspace is resolvable* [**2**].

(iv) *A space is a* partition space *if its open sets form a Boolean algebra* [**44**, Example 5].

We note that many texts require that an extremally disconnected space satisfy the additional condition of being Hausdorff ([**7**, Exercise I.11.21] for example); our definition however agrees with [**27**]. We use the notation 'OU-space' to abbreviate 'open unresolvable space'.

Theorem 2.3. (i) *A space satisfies $ibi = bi$ if and only if it is extremally disconnected.*

(ii) *A space satisfies $ibi = ib$ if and only if it is an OU-space if and only if each of its dense sets has dense interior.*

(iii) *A space satisfies $ib = bi$ if and only if it is an extremally disconnected OU-space.*

(iv) *A space satisfies $ib = b$ if and only if it is a partition space if and only if it its open sets are clopen (that is, simultaneously closed and open).*

(v) *A space satisfies $i = b$ if and only if it is discrete.*

Proof. All except part (ii) are trivial or have been established above. In [**2**], Aull shows that OU-spaces are exactly spaces in which every dense set has dense interior and that spaces whose dense sets have dense interiors satisfy $ibi = bi$. To prove the converse, let $(X, \mathcal{T})$ satisfy $ibi = bi$ and let A be dense in X. Therefore $ib(A) = i(X) = X$. Since $ibi(A) = ib(A)$ we must have $bi(A) \supseteq X$, which implies that $i(A)$ is dense in X also. □

The following result is stated in [**44**, Example 5, part 1]. The proof of the last statement is not quite trivial.

Proposition 2.4. *Let X be a set and θ be an equivalence relation on X. Then the topology on X whose basis consists of the equivalence classes of X (along with the empty set) is a partition space. Furthermore, every partition space arises in this way.*

We note that the 6 different Kuratowski monoids of non-empty topological spaces also admit a natural partial order given in Figure 2.3. Here $\mathbf{M}_1 \leq \mathbf{M}_2$ is there is a monoid homomorphism from $\mathbf{M}_2$ onto $\mathbf{M}_1$.

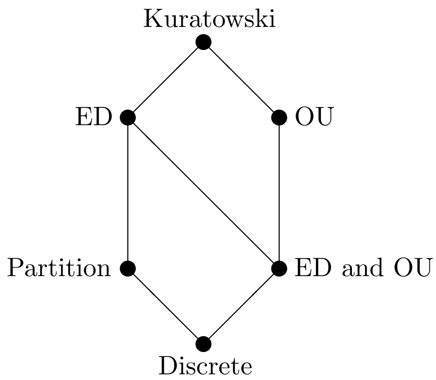


FIGURE 2.3. The six different Kuratowski monoids for topological spaces.

2.2. Classifying spaces by their k-number. The most indepth investigation along these lines is given by Aull in [**2**]. Aull is primarily concerned with Hausdorff spaces and with this restriction gives a description for most of the possible k-numbers (Theorem 2.6 below).

To begin with, note that the only possible k-numbers of a set in a space are the positive even numbers less than or equal to 14. Indeed let f and g be even operators. Then $f(A) = g(A)$ if and only if $af(A) = ag(A)$, so that $|\{f(A) : f \text{ is even}\}| = |\{g(A) : g \text{ is odd}\}|$. Hence, if no odd operator agrees on A with an even operator, then the k-number of A is even. In the contrary case, let $o_1(A) = o_2(A)$ where o_1 is even and o_2 is odd. Now i is the smallest even operator and so ai is the largest odd operator. Therefore we have $i(A) \subseteq o_1(A) = o_2(A) \subseteq ai(A)$. That is, $i(A)$ is contained in its complement and therefore is empty. A dual argument shows that A is dense in X and so the distinct sets obtainable from A using Kuratowski operators are A, $a(A)$, $\varnothing$, X; so in this case also the k-number of A is even.

To state the Aull result we shall have need of the concept of a *door space* introduced in [**27**]: a space in which every set is either open or closed. Any space with at most one non-isolated point is a door space; for Hausdorff spaces this is if and only if [**27**, p. 76]. A useful lemma is the following.

Lemma 2.5. [**2**] *Every door space is an OU space.*

Aull's results are stated in terms of the maximum number of sets obtainable from an arbitrary subset of a space using interior and closure. This is usually equal to half the number obtainable using closure and complement, a rule that can only fail if there is an even operator and a closed operator that act identically on A. As discussed at the start of this subsection, this implies that A is dense and has empty interior, in which case $k(A) = 4$. A space containing a neutral set cannot be a door space and hence has at least two non-isolated points. As we show below (Lemma 2.7 and associated discussion), in Hausdorff spaces this implies a k-number of at least 8. Thus no new Hausdorff spaces with k-number 4 are possible and the Aull result continues to hold under our notation.

Theorem 2.6. [**2**] *Let $(X, \mathfrak{T})$ be a Hausdorff space with k-number n.*
- (i) $n = 2$ *if and only if* $(X, \mathfrak{T})$ *is discrete.*
- (ii) $n = 4$ *if and only if either* $(X, \mathfrak{T})$ *is a non-discrete door space.*
- (iii) $n = 8$ *if* $(X, \mathfrak{T})$ *is an extremally disconnected OU space that is not a door space.*
- (iv) $n = 10$ *only if* $(X, \mathfrak{T})$ *is an extremally disconnected but not OU space or an OU but not extremally disconnected space.*
- (v) $n = 14$ *if and only if* $(X, \mathfrak{T})$ *is neither extremally disconnected nor OU.*
- (vi) $n \neq 6, 12$.

If $(X, \mathfrak{T})$ has no isolated points then all of the possibilities are if and only if.

We note that Theorem 2 of [**2**] states that a space $(X, \mathfrak{T})$ without isolated points has k-number at most 6 if and only if X does not contain two disjoint open sets and that a space with two disjoint open sets each containing non-isolated points has k-number at least 8. This theorem (on which the $n \neq 6$ part of Theorem 2.6 (vi) depends) does not assume the Hausdorff condition but without it, the second statement is in fact false (the first statement holds). Indeed, let θ be an equivalence on a set X with more than one non-singleton equivalence class. In the corresponding partition space, there are two non-isolated points contained in disjoint open sets while the k-number of a partition space is at most 6.

With the Hausdorff condition assumed, Theorem 2 of [**2**] is true, as we now show. First, let a point x be called *weakly isolated* if $ib(\{x\})$ is non-empty.

Lemma 2.7. *Let $(X, \mathfrak{T})$ be a space with two disjoint open sets H_1 and H_2, each containing non-weakly isolated points, x_1 and x_2 respectively. Then the k-number of $(X, \mathfrak{T})$ is at least* 8.

Proof. We show that the set $M = (H_1 \backslash \{x_1\}) \cup \{x_2\}$ has k-number at least 8. Note that x_1 and x_2 cannot be isolated (in fact our proof only uses the assumption that x_1 is not isolated and x_2 is not weakly isolated). We begin by showing that $x_1, x_2 \in b(M)$.

If V is an open set containing x_1 then $x_1 \in V \cap H_1 \in \mathfrak{T}$. Since x_1 is not isolated, $V \cap H_1 \neq \{x_1\}$, so $M \cap V = (M \cap V) \backslash \{x_1\} \supseteq ((H_1 \backslash \{x_1\}) \cap V) \backslash \{x_1\} = (H_1 \cap V) \backslash \{x_1\} \neq \varnothing$. Thus, x_1 is a limit point of M, and $x_1, x_2 \in b(M)$ as required.

Now $x_1 \in H_1 \subseteq b(M)$ so that $x_1 \in ib(M)$. We show that $x_2 \notin ib(M)$. Now $b(M) = b(H_1 \backslash \{x_1\}) \cup b(\{x_2\}) \subseteq b(H_1) \cup b(\{x_2\})$. Let V be an open set containing x_2. So $U := V \cap H_2$ is an open set containing x_2. Since H_1 and H_2 are disjoint, so are $b(H_1)$ and H_2, whence $U \cap b(M) \subseteq U \cap (b(H_1) \cup b(\{x_2\})) = U \cap b(\{x_2\})$ and $x_2 \in U \cap b(\{x_2\})$. Now $ib(\{x_2\}) = \varnothing$ and so there is a point in U not in $b(\{x_2\})$ and therefore not in $b(M)$ either. Hence x_2 is a limit point of $ab(M)$ and $x_2 \notin ib(M)$.

Finally, we note that since $x_2 \notin ib(M)$, we have $x_1, x_2 \notin i(M)$. These facts and the definition of M ensure that M, $i(M)$, $ib(M)$ and $b(M)$ are all distinct. □

Now to show that no Hausdorff space has k-number 6. First note that since singleton sets are closed in a Hausdorff space, we have for any point x, $ib(\{x\}) = i(\{x\})$ and so the notion of weakly isolated coincides with that of being isolated. Now let $(X, \mathcal{T})$ be a Hausdorff space. If $(X, \mathcal{T})$ contains at most one non-isolated point then it is a door space and has k-number 2 or 4. Otherwise it has at least two points that are not isolated and hence, by the Hausdorff condition, not weakly isolated. So Lemma 2.7 can be applied, giving a k-number of at least 8.

Part (i) of this theorem obviously does not depend on the Hausdorff condition, however it is easy to find examples of finite spaces contradicting all the other statements. The following lemma extends part (ii) of Theorem 2.6 to include non-Hausdorff spaces.

Lemma 2.8. *A space $(X, \mathcal{T})$ has k-number 4 if and only if it is a non-discrete door space or $\mathcal{T} \backslash \{\varnothing\}$ is a filter in the power set of X.*

Proof. Let us say that $(X, \mathcal{T})$ has k-number 4 but is not a door space. Therefore there is a neutral set A. Then A is both dense and co-dense, for if $i(A)$ is non-empty or $b(A)$ is not equal to X, then A, $i(A)$, $b(A)$ and their complements are distinct and $k(A) \geq 6$. This also shows that any set containing a non-empty open subset is either open or closed. We now show that no disjoint pair of non-empty open sets exists. This will in addition show that any set containing an open subset is open and since open sets are closed under finite intersections, $\mathcal{T} \backslash \{\varnothing\}$ will be a filter.

Assume that U and V are disjoint non-empty open sets in $\mathcal{T}$. Now $A \cup U$ is not equal to X since $(A \cup U) \cap V = A \cap V \subsetneq V$ (because A is co-dense). Therefore $A \cup U$ is not closed because $b(A) = X$. As $A \cup U \supset U$, it follows that $A \cup U$ is open. But then $(A \cup U) \cap V = A \cap V$ is open and non-empty, contradicting the fact that A has empty interior. It now follows by contradiction that no disjoint non-empty open sets exist, which completes the proof that $\mathcal{T} \backslash \{\varnothing\}$ is a filter.

For the reverse implication, we may assume that $\mathcal{T} \backslash \{\varnothing\}$ is a filter (the door space case is proved in [**2**]) and so there are no disjoint non-empty open sets. Then the closure of any non-empty open set is X, the interior of any closed set unequal to X is empty and no neutral set contains a non-empty open subset, so all neutral sets are both dense and co-dense. Thus the possible k-numbers of sets are 2 (for $\varnothing$ and X) and 4. □

We note in passing that if $\mathcal{T} \backslash \{\varnothing\}$ is a filter, or more generally if there are no disjoint non-empty open sets, then $(X, \mathcal{T})$ is extremally disconnected.

The following lemma sheds some light on spaces with k-number 6 and is useful below.

Lemma 2.9. *Let $(X, \mathfrak{T})$ be an extremally disconnected OU space. Then $k((X, \mathfrak{T})) = 6$ if and only if $(X, \mathfrak{T})$ is not a door space and every neutral set has clopen interior or closure.*

Proof. Let $(X, \mathfrak{T})$ be extremally disconnected OU space with k-number 6. Theorem 2.6 (i) and (ii), and Lemma 2.8 show that $(X, \mathfrak{T})$ is not a door space. Say there exists a neutral set $A \subseteq X$ with $b(A)$ and $i(A)$ not clopen. Since the closure of an open set in an extremally disconnected space is clopen, it follows that there is a clopen set $B \neq i(A), b(A), A$ such that $bi(A) = B$. This contradicts $k((X, \mathfrak{T})) = 6$.

Now let $(X, \mathfrak{T})$ be an extremally disconnected OU space that is not a door space and is such that every neutral set has clopen interior or closure. Let $A \subseteq X$. If A is open or closed Figure 2.1 shows that A has a k-number of at most 4. If A is neutral and $i(A)$ is clopen then Figure 2.1 shows that there are precisely three sets obtainable from A by taking closures and interiors, giving a k-number of 6 for A. A dual argument holds when $b(A)$ is clopen. □

It is also easy to see that partition space has k-number 6 if and only if it is neither discrete nor anti-discrete.

A further result relating spaces to their k-number is given in [**4**] and [**22**] where it is shown that $k((X, \mathfrak{T})) = 14$ implies that $|X| \geq 7$; an example of a full Kuratowski space on seven points is also given in both papers and in [**22**] it is shown that there are exactly 5 non-homeomorphic spaces on 7 points with k number 14. These results are extended in [**1**] where it shown that $k((X, \mathfrak{T})) \leq 2|X|$ for all $|X| \leq 7$ (and that this bound can be achieved in every case). Furthermore, for all $n \leq 7$, the number of non-homeomorphic n-point spaces with k-number $2n$ is determined.

2.3. The k-numbers of sets in topological spaces. The earliest work (other than that of Kuratowski) concerning the possible behaviour of the topological operators on individual sets appears to be that of Levine [**31**] and Chapman [**11**]. Levine gives characterisations of subsets of spaces on which the interior and closure operators commute. In [**11**], two functions are considered which associate with each subset A of X, a positive integer less than 9 related to which of a certain collection of the 14 Kuratowski operators act differently on A; the possible values of these functions is examined. These functions, called *line functions* are then abstracted and it is shown that an arbitrary set X and with a pair of line functions acting on the subsets of X determines a topology on X.

In [**12**] Chapman presents a deeper investigation into which Kuratowski operators can act differently on subsets of topological spaces. For every pair (o_1, o_2) of the 7 operators in Figure 1.1, a description is given of those sets A for which o_1 and o_2 act identically on A. The commutativity of closure and interior, as investigated in [**31**], is a particular case. A description of typical flavour is that $A = bi(A)$ if and only if A is closed and is the union of an open set and a set with empty interior in the subspace on A.

Papers [**30**] and [**39**] primarily concern the characterisation of 14-sets. The main theorem in [**30**] gives five independent conditions on a set A, which together are equivalent to A being a 14-set. This is then used to show that for every 14-set $A \subseteq \mathbb{R}$ with the usual topology there is an interval I in which both A and $a(A)$ are dense (this solved a problem posed by Baron [**3**]). In [**39**] there is another characterisation of 14-sets under the additional assumption that the topological space be connected. Several variations of the notion of boundary are introduced and it is shown that in a connected space $(X, \mathfrak{T})$, a subset $A \subseteq X$ is a 14-set if and only if certain boundaries of A are non-empty.

A given subset A of a topological space determines an equivalence $\equiv_A$ on the set of Kuratowski operators, where $o_1 \equiv_A o_2$ if $o_1(A) = o_2(A)$. This equivalence is clearly stable under composition from the left by operators and respects the partial order.

Theorem 2.10. *The action of the even Kuratowski operators on an individual set falls into one of the 30 possibilities given in Table 2.1. If the set is not dense with empty interior, the action of the odd operators produces the same number of sets: the complement of those produced by the even operators. If a set A is dense with empty interior, then the distinct sets produced by all Kuratowski operators are A, $a(A)$, $\varnothing$, X.*

Proof. It follows from the argument that starts Subsection 2.2 that the restriction of our arguments to the even Kuratowski operators results in only one possible case being overlooked. This situation is when a set is dense and has empty interior—under the even Kuratowski operators this is indistinguishable from an arbitrary set with clopen closure and clopen interior (which may not be dense and may not have empty interior). All of the remaining arguments will concern only the even operators.

We are looking for possible collapses under relations $\equiv_A$ of the diagram in Figure 1.1. In the discussion that follows we shall deal with a general set A, but for notational simplicity we shall write $\equiv$ in place of $\equiv_A$.

First note that if $bi \equiv bib$ then $ibi \equiv ibib = ib$, while if $ib \equiv ibi$, then $bib \equiv bibi = bi$. Also if $bi \equiv ib$, then $bi = bbi \equiv bib$ and $ib = iib \equiv ibi$. Thus as far as ib, bi, ibi and bib are concerned, there are at most the following possibilities:

(i) no two are equivalent;
(ii) all are equivalent;
(iii) $bi \equiv ibi$ and no others are equivalent;
(iv) $ib \equiv bib$ and no others are equivalent;
(v) $bi \equiv ibi$, $ib \equiv bib$ and no others are equivalent;
(vi) $bi \equiv bib$, $ib \equiv ibi$ and no others are equivalent.

Meanwhile, there are at most the following possibilities for b and i:

(vii) Neither is equivalent to any other operator;
(viii) $ibi \equiv i$;
(ix) $bib \equiv b$;
(x) $ibi \equiv i$ and $bib \equiv b$.

If $b \equiv \mathsf{id}$ but $i \not\equiv \mathsf{id}$, then $ib \equiv i\,\mathsf{id} = i$ so (by order preservation), $ib \equiv ibi \equiv i$ and also (by left stability), $bib \equiv bi$. As $i \not\equiv \mathsf{id}$, all of this is consistent with no more than the following:

(xi) no further equivalences;
(xii) $bib \equiv b$ and there are no other equivalences;
(xiii) $bib \equiv ib$ and there are no other equivalences.

These last statements have corresponding duals, (xiv), (xv) and (xvi) if $i \equiv \mathsf{id}$ and $b \not\equiv \mathsf{id}$.

Thus there are at most 30 different ways in which the even operators can be equivalent on a set:

- one of (i)–(vi) and one of (vii)–(x);
- one of (xi)–(xvi).

Note that the combination (ii) with (x) gives us $i \equiv ibi \equiv bib \equiv b$, so as $i \leq \mathsf{id} \leq b$, all even operators coincide. This happens for clopen sets only. The other "combined" conditions apply to neutral sets, while (xi)–(xiii) (resp. (xiv)–(xvi)) apply to sets which are closed but not open (resp. open but not closed).

Most of the equivalence patterns for even operators can be realised on a subset of $\mathbb{R}$, however some are clearly not realisable on any connected space. Indeed for a connected space on X with $A \subseteq X$, if $o(A)$ is clopen for some even operator o then either $b(A)$ is clopen or $i(A)$ is clopen. If $o_1 \not\equiv_A o_2$ are two even operators for which $o_1(A)$ and $o_2(A)$ are clopen, then $i(A) = \varnothing$ and $b(A) = X$. There are eight equivalences of this type, and we can realise each using a subset of the Cantor space; Table 2.1 gives examples for all possibilities.

If no odd and even operators agree on a set A, then $k(A) = 2|\{o(A) : o \text{ is even}\}|$ while in the contrary case, as noted in Section 2.2, A is dense with empty interior and $k(A) = 4$. □

The k-numbers of a set giving rise to any of the equivalence patterns are given in Table 2.1. The final column gives an example with the corresponding pattern. Here we let $A = (0, 1)$, $B = (1, 2)$, $C = \{3\}$, $D = [4, 5]$ and $E = \mathbb{Q} \cap [6, 7]$, taken as subsets of the reals given the usual topology. For those equivalences not realised by any subset of a connected space, we use subsets of the Cantor space $\mathcal{C}$. Let $F = [0, 1/4) \cap \mathcal{C}$, $G = \{\frac{n}{3^m} : m \in \mathbb{N} \text{ and } 0 < n \leq 3^{m-1}\} \cap \mathcal{C}$ (these are the end points of excluded intervals of $\mathcal{C}$ in the interval $[0, 1/3]$), $H = \{2/3\}$ and $I = \mathcal{C} \cap (20/27, 21/27)$ and $J = \mathcal{C} \cap (11/12, 1]$. We leave it to the reader to establish the following facts: $b(G) = \mathcal{C} \cap [0, 1/3]$; $i(G) = \varnothing$; $b(F) = [0, 1/4]$; $ib(F) = [0, 1/4)$; $b(J) = [11/12, 1]$; $i(J) = (11/12, 1]$; $i(F \cup G) = F$; and $b(F \cup G) = [0, 1/3] \cap \mathcal{C}$.

The example of a 14-set in the Cantor space gives us the following result.

Proposition 2.11. *The Cantor space is a full Kuratowski space.*

This result also follows from Theorem 5.10 in [**32**]. Aull comments that the rationals given the usual subspace topology from the reals is a full Kuratowski space.

3. Stability Under Topological Constructions

We now examine what kind of properties of the Kuratowski operators are preserved under the basic topological constructions. The main results in this section are all new and so we begin with a subsection establishing some useful tools.

3.1. Kuratowski acts and stone duality. Recall that if $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are spaces then a map $f : X \to Y$ is called an *open* (*closed*) map if it maps open sets (closed sets) to open sets (closed sets).

Lemma 3.1. *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and $f : X \to Y$. If f is an open and continuous map then $f^{-1}(b(A)) = b(f^{-1}(A))$ for every $A \subseteq Y$.*

Collapse:	Cannot occur in spaces of type:	k:	Example:
$\varnothing$	ED, OU	14	$A \cup B \cup C \cup E$ in $\mathbb{R}$, or $F \cup G \cup H \cup I \cup J$ in $\mathcal{C}$
$bi = ibi$	ED, OU, conn.	12	$F \cup G \cup H \cup I$ in $\mathcal{C}$
$ib = bib$	ED, OU, conn.	12	$a(F \cup G \cup H \cup I)$ in $\mathcal{C}$
$bib = b$	ED, OU	12	$A \cup B \cup E$ in $\mathbb{R}$
$ibi = i$	ED, OU	12	$A \cup C \cup E$ in $\mathbb{R}$
$ib = ibi$, $bi = bib$	ED	10	$A \cup B \cup C$ in $\mathbb{R}$
$ib = bib$, $bi = ibi$	OU, Part., conn.	10	$G \cup H \cup I$ in $\mathcal{C}$
$ib = bib$, $ibi = i$	ED, OU, conn.	10	$F \cup G \cup H$ in $\mathcal{C}$
$bi = ibi$, $bib = b$	ED, OU, conn.	10	$a(F \cup G \cup H)$ in $\mathcal{C}$
$bi = ibi = i$	ED, OU	10	$C \cup E$ in $\mathbb{R}$
$ib = bib = b$	ED, OU	10	$a(C \cup E)$ in $\mathbb{R}$
$bib = b$, $ibi = i$	ED, OU	10	$A \cup E$ in $\mathbb{R}$
$ibi = bi = ib = bib$	Part., conn.	8	$H \cup I$ in $\mathcal{C}$
$ib = ibi = i$, $bi = bib$	ED	8	$A \cup C \cup D$ in $\mathbb{R}$
$ib = bib$, $bi = ibi = i$	OU, Part., conn.	8	$G \cup H$ in $\mathcal{C}$
$ib = ibi$, $bi = bib = b$	ED	8	$A \cup B \cup D$ in $\mathbb{R}$
$ib = bib = b$, $bi = ibi$	OU, Part., conn.	8	$G \cup I$ in $\mathcal{C}$
$bi = ibi = i$, $bib = b$	ED, OU	8	$\mathbb{Q} \backslash A$ in $\mathbb{R}$
$ib = bib = b$, $ibi = i$	ED, OU	8	$\mathbb{Q} \cup A$ in $\mathbb{R}$
$ibi = bi = ib = bib = i$	Part.	6	$\{1/n : n \in \mathbb{N}\}$ in $\mathbb{R}$
$ibi = bi = ib = bib = b$	Part.	6	$a(\{1/n : n \in \mathbb{N}\})$ in $\mathbb{R}$
$ib = ibi = i$, $bi = bib = b$	ED	6	$A \cup D$ in $\mathbb{R}$
$\mathsf{id} = b$, $ib = ibi = i$, $bi = bib$	ED	6	$C \cup D$ in $\mathbb{R}$
$\mathsf{id} = i$, $bi = bib = b$, $ib = ibi$	ED	6	$A \cup B$ in $\mathbb{R}$
$ib = bib = b$, $bi = ibi = i$	OU	4,6	$\mathbb{Q}$ in $\mathbb{R}$
$\mathsf{id} = bi = bib = b$, $ib = ibi = i$	ED	4	D in $\mathbb{R}$
$\mathsf{id} = ib = ibi = i$, $bi = bib = b$	ED	4	A in $\mathbb{R}$
$\mathsf{id} = b$, $ibi = bi = ib = bib = i$	Part.	4	C in $\mathbb{R}$
$\mathsf{id} = i$, $ibi = bi = ib = bib = b$	Part.	4	$a(C)$ in $\mathbb{R}$
$ibi = bi = ib = bib = b = i = \mathsf{id}$	–	2	$\varnothing$ in $\mathbb{R}$

TABLE 2.1. All 30 kinds of sets in a topological space.

Proof. It is well known that a map f is continuous if and only $f^{-1}(b(A)) \supseteq b(f^{-1}(A))$ for every $A \subseteq Y$ [**27**, Theorem 3.1]. We now show that if f is open then $f^{-1}(b(A)) \subseteq b(f^{-1}(A))$. Let $x \in f^{-1}(b(A))$ so that $f(x) \in b(A)$. If $f(x) \in A$ then $x \in f^{-1}(A) \subseteq b(f^{-1}(A))$. Now assume that $f(x) \notin A$; that is, that $f(x)$ is a limit point of A. For every neighbourhood B of x we have $f(x) \in f(B)$ and since $f(x)$ is a limit point of A and $f(B)$ is open, there is an element $f(y) \in f(B) \cap A$, where $y \in B$. Since $f(x) \neq f(y) \in A$ we have $x \neq y \in B \cap f^{-1}(A)$. That is, for any neighbourhood B of x there is an element $y \in B \cap f^{-1}(A)$ and therefore $x \in b(f^{-1}(A))$. Hence $f^{-1}(b(A)) \subseteq b(f^{-1}(A))$ as required. □

Let $\mathbf{K}$ denote the Kuratowski monoid of operators and let $\mathcal{P}(X)$ denote the power set of a set X. If $(X, \mathfrak{T})$ is a topological space, then the monoid $\mathbf{K}$ acts on $\mathcal{P}(X)$ in an obvious way, giving it the structure of a *Kuratowski-act* (denoted $\mathcal{P}(\mathbf{X})_{\mathcal{T}}$). A homomorphism between Kuratowski acts $\mathcal{P}(\mathbf{X})_{\mathcal{T}}$ and $\mathcal{P}(\mathbf{Y})_{\mathcal{U}}$ will be a map $\phi : \mathcal{P}(X) \to \mathcal{P}(Y)$ satisfying $\phi(o(A)) = o(\phi(A))$ for each Kuratowski operator o.

Lemma 3.2. *Let $(X, \mathfrak{T})$ and $(Y, \mathfrak{U})$ be topological spaces and $f : X \to Y$ an open, continuous map. Then f^{-1} is a homomorphism of Kuratowski-acts $f^{-1} : \mathcal{P}(\mathbf{Y})_{\mathfrak{U}} \to \mathcal{P}(\mathbf{X})_{\mathfrak{T}}$.*

Proof. By Lemma 3.1 it remains to show that $f^{-1}(a(A)) = a(f^{-1}(A))$ for every $A \subseteq Y$. This follows immediately since $f^{-1}(A) \cup f^{-1}(a(A)) = f^{-1}(Y) = X$ while $f^{-1}(A) \cap f^{-1}(a(A)) = \varnothing$. □

Let TOPO denote the category of topological spaces with continuous open maps as arrows (which are clearly closed under composition) and KACT denote the category of Kuratowski-acts with homomorphisms as arrows. We define $\Phi : \mathsf{TOPO} \to \mathsf{KACT}$ by $(X, \mathfrak{T}) \mapsto \mathcal{P}(\mathbf{X})_{\mathfrak{T}}$ and $f \mapsto f^{-1}$ for continuous open maps f. (Because we wish f^{-1} to be an arrow in KACT, we here stress that f^{-1} is interpreted as a map between the power sets of spaces, rather than a map between spaces themselves.) In view of Lemma 3.2, it is easily verified that Φ is a contravariant functor from TOPO to KACT.

Now recall that the notion of a topological sum of spaces. For a family $\{X_i : i \in I\}$ of sets, we let $\dot\cup_{i \in I} X_i$ denote the disjoint union $\cup_{i \in I} X_i \times \{i\}$. If no confusion arises we drop the second coordinate.

Definition 3.3. *Let $\{(X_i, \mathfrak{T}_i) : i \in I\}$ be a family of spaces. The* sum space *$\sum_{i \in I}(X_i, \mathfrak{T}_i)$ is the space on $\dot\cup_{i \in I} X_i$ with basis $\{S \times \{i\} : S \in \mathfrak{T}_i$ for some $i \in I\}$.*

Proposition 3.4. (i) $\Phi(\sum_i (X_i, \mathfrak{T}_i)) \cong \prod_i \Phi(X_i, \mathfrak{T}_i)$.

(ii) *If $f : (X, \mathfrak{T}) \to (Y, \mathfrak{U})$ is an injective continuous open map, then $\Phi(X, \mathfrak{T})$ is a homomorphic image of $\Phi(Y, \mathfrak{U})$.*

(iii) *If $f : (X, \mathfrak{T}) \to (Y, \mathfrak{U})$ is a surjective continuous open map, then $\Phi(Y, \mathfrak{U})$ embeds into $\Phi(X, \mathfrak{T})$.*

Proof. (i). Let $(X, \mathfrak{T})$ denote $\sum_i (X_i, \mathfrak{T}_i)$. The map $A \mapsto (A \cap X_i)_i$ is easily seen to be a bijection between $\mathcal{P}(X)$ and the cartesian product $\prod_i \mathcal{P}(X_i)$. Moreover, it is easily seen that for $A \subseteq X$ we have $b(A) = \dot\cup_i b(A \cap X_i) \mapsto (b(A \cap X_i))_i$ and similarly for complements. Hence this bijection is an isomorphism between $\mathcal{P}(\mathbf{X})_{\mathfrak{T}}$ and the direct product of Kuratowski-acts $\prod_i \mathcal{P}(\mathbf{X_i})_{\mathfrak{T}_i}$.

(ii). By Lemma 3.2 we need only show the associated homomorphism $f^{-1} : \mathcal{P}(\mathbf{Y})_{\mathfrak{U}} \to \mathcal{P}(\mathbf{X})_{\mathfrak{T}}$ is surjective. This holds because the injectivity of f implies $A = f^{-1}(f(A))$ for every $A \in \mathcal{P}(X)$.

(iii). As in (ii), $f^{-1} : \mathcal{P}(\mathbf{Y})_{\mathfrak{U}} \to \mathcal{P}(\mathbf{X})_{\mathfrak{T}}$ is a homomorphism. Let $A \neq B$ be subsets of Y. Without loss of generality there is $y \in A \backslash B$ and then there exists $x \in X$ with $f(x) = y$. So $f(x) \in f^{-1}(A) \backslash f^{-1}(B)$, showing that f^{-1} is injective. □

The following easy lemma demonstrates some of the implications of Proposition 3.4.

Lemma 3.5. *Let $f : \Phi(X, \mathcal{T}) \to \Phi(Y, \mathcal{U})$ be a homomorphism of Kuratowski acts.*

(i) *If f is injective then the Kuratowski monoid of $\Phi(X, \mathcal{T})$ is a homomorphic image of that of $\Phi(Y, \mathcal{U})$.*

(ii) *If f is surjective then the Kuratowski monoid of $\Phi(Y, \mathcal{U})$ is a homomorphic image of that of $\Phi(X, \mathcal{T})$.*

Proof. If f is injective then whenever two Kuratowski operators act differently on $(X, \mathcal{T})$, there are corresponding subsets of Y on which the same operators act differently. Hence the Kuratowski monoid of $(Y, \mathcal{U})$ has fewer identifications than that of $(X, \mathcal{T})$.

If f is surjective then Kuratowski operators acting identically on $\Phi(X, \mathcal{T})$ also act identically on $\Phi(Y, \mathcal{U})$, so the Kuratowski monoid of $(X, \mathcal{T})$ has fewer identifications than that of $(Y, \mathcal{U})$. □

In the coming subsections we will also find it useful to recall some basic facts of Stone duality [**45**]. We direct the reader to Halmos [**20**] for an easy introduction and proofs of the statements below. (Note that our notation is different to that in [**20**].)

For a Boolean algebra B, let B^* denote the set of all ultrafilters of B given a topological basis consisting of those sets of the form $U_x := \{U : U \in B^* \text{ and } x \in U\}$ for each $x \in B$. This space is compact and Hausdorff and the basis sets can be seen to be clopen (if x' is the Boolean complement of x in B then $U_{x'}$ is the complement in B^* of U_x). Conversely, with any compact Hausdorff space X with a basis of clopen sets (a so-called *Boolean space*), one may associate the Boolean algebra of clopen subsets, denoted X^*. These associations turn out to constitute a dual equivalence—Stone duality—between the category of Boolean spaces (with continuous maps) and of Boolean algebras (with homomorphisms). Thus we get $(B^*)^*$ is a Boolean algebra isomorphic to the Boolean algebra B and $(X^*)^*$ is a Boolean space homeomorphic to the Boolean space X. The dual of a homomorphism $f : B_1 \to B_2$ between Boolean algebras is the (continuous) map $f^* : B_2^* \to B_1^*$ that assigns each ultrafilter contained in B_2^*, its preimage under f (an ultrafilter of B_1 and hence an element of B_1^*). Conversely, the dual of a continuous map $g : X_1 \to X_2$ between Boolean spaces is the map $g^* : X_2^* \to X_1^*$ that assigns each clopen set of X_2, its preimage under g (which is also clopen because g is continuous). Injective maps become surjections, and surjections become injections.

Our use of Stone duality will be based on the following two facts.

Proposition 3.6. (i) *A Boolean space is extremally disconnected if and only its dual Boolean algebra is complete (that is, every subset has a supremum).*

(ii) *Any continuous morphism between compact Hausdorff spaces is a closed map.*

3.2. Subspaces. Proposition 3.4 is easily used to show that the Kuratowski operators are well behaved with respect to the taking of open subspaces.

Theorem 3.7. *Let $(X, \mathcal{T})$ be a topological space and $A \in \mathcal{T}$. Then the subspace on A has K-number at most that of $(X, \mathcal{T})$. More specifically, the Kuratowski monoid of $(X, \mathcal{T})$ is ordered above or equal (in Figure 2.3) to that of the open subspace on A.*

Proof. Let $(A, \mathcal{T}_A)$ denote the subspace of $(X, \mathcal{T})$ on A, and let $\iota : A \to X$ denote the inclusion map. We show that ι is a continuous open map from the subspace on A. Let B be open in X. Then $\iota^{-1}(B) = \{a \in A : \iota(a) \in B\} = A \cap B$ which is open in the subspace on A by definition. Also if B is an open set in the subspace on A then there is an open set $C \in \mathcal{T}$ such that $A \cap C = B$. But A is open and hence $B = A \cap C \in \mathcal{T}$. Therefore ι is an open continuous map that is clearly injective. So Proposition 3.4 (ii) implies that $\iota^{-1} : \mathcal{P}(\mathbf{X})_{\mathcal{T}} \to \mathcal{P}(\mathbf{A})_{\mathcal{T}_A}$ is a surjective homomorphism of Kuratowski-acts. By Lemma 3.5 the Kuratowski monoid of $(A, \mathcal{T}_A)$ is a homomorphic image of that of $(X, \mathcal{T})$. □

In contrast with this we now show that most of the possible identifications of Kuratowski operators are not preserved under taking arbitrary—even closed—subspaces. We begin with an elementary construction.

Definition 3.8. *If $(X, \mathcal{T})$ is a topological space and $x \notin X$ then we let $(X \cup \{x\}, \mathcal{T}_x)$ denote the space on $X \cup \{x\}$ with topology given by $\{\varnothing\} \cup \{S \cup \{x\} : S \in \mathcal{T}\}$. This is called the* closed extension *of* $(X, \mathcal{T})$ [**44**].

Theorem 3.9. *Every space is a closed subspace of an extremally disconnected OU-space with k-number at most* 6.

Proof. The proof of the first statement will follow from the fact that the closed extension of a space is extremally disconnected and OU. First note that the dense sets in $(X \cup \{x\}, \mathcal{T}_x)$ are exactly those containing x. Therefore all non-empty open sets are dense and this space is extremally disconnected. Second, the interior of a set containing x also contains x and so dense sets have dense interiors implying that the space is OU as well. Furthermore, a set not containing x has empty, whence clopen, interior while the closure of a set containing x is the whole space, again clopen. (The closed sets are the closed sets of X together with $X \cup \{x\}$.) By Lemma 2.9 the k-number of $(X \cup \{x\}, \mathcal{T}_x)$ is equal to 6, or is a door space (when $(X, \mathcal{T})$ is discrete). □

In terms of the 6 possible Kuratowski monoids, this result cannot be further improved.

Proposition 3.10. *A subspace of a partition space is a partition space or is discrete.*

Proof. Note that if A is clopen in a space $(X, \mathcal{T})$ and $Y \subseteq X$, then $Y \cap A$ is clopen in the subspace on Y. Since all open sets in a partition space are clopen, the same is true for their subspaces. □

Behaviour similar to that exhibited in Theorem 3.9 can also be achieved amongst more familiar spaces. For example, it is known that a homomorphic image of the (complete) Boolean algebra $\mathcal{P}(S)$ of all subsets of a set S, need not be complete (see [**20**, Chapter 25] for example). Let $f : \mathcal{P}(S) \to B$ be such a case. By taking the Stone dual, we obtain an injective continuous map from B^* into $(\mathcal{P}(S))^*$. By Proposition 3.6, we find that $(\mathcal{P}(S))^*$ is extremally disconnected while B^* is not extremally disconnected and yet is homeomorphic to a closed subspace of $(\mathcal{P}(S))^*$. We note that $(\mathcal{P}(S))^*$ is known to be homeomorphic to the Stone-Čech compactification of the discrete space on S.

The closed extension of a space corresponds to adjoining a new point to every open set. Now let us consider the space obtained by adjoining a new point to every closed set; that is, for a space $(X, \mathcal{T})$ with $x \notin X$, let $(X \cup \{x\}, \mathcal{T}^x)$ denote the space on $X \cup \{x\}$ with topology $\mathcal{T} \cup \{X \cup \{x\}\}$. This construction is called the *open extension* of a space [**44**]. Because X is open in $(X \cup \{x\}, \mathcal{T}^x)$, Theorem 3.7 will show that $K(X \cup \{x\}, \mathcal{T}^x) \geq K(X, \mathcal{T})$. In fact we have the following.

Proposition 3.11. *Let $(X, \mathcal{T})$ be a space and let $x \notin X$. Then the open extension of $(X, \mathcal{T})$ is:*

(i) *extremally disconnected and OU with K-number 8 if and only if $(X, \mathcal{T})$ is OU and contains no disjoint non-empty open subsets;*
(ii) *OU with K-number 10 if and only if $(X, \mathcal{T})$ is OU and contains two disjoint open subsets;*
(iii) *extremally disconnected with K-number 10 if and only if $(X, \mathcal{T})$ contains no disjoint open sets and is not OU.*

If none of these conditions hold then $(X, \mathcal{T})$ is a Kuratowski space.

Proof. In this proof we let $(X, \mathcal{T})$ be an arbitrary space not containing x and with interior and closure operators called i_1 and b_1. We denote the corresponding operators in the space $(X \cup \{x\}, \mathcal{T}^x)$ by i and b. Note that every non-empty closed set in $X \cup \{x\}$ contains x, and if $A \subseteq X$ then $i(A) = i_1(A)$ and $b(A) = b_1(A) \cup \{x\}$. We prove three facts and deduce the result from these.

(1) If $(X, \mathcal{T})$ has no disjoint non-empty open sets, then all non-empty open sets are dense. If $\varnothing \subsetneq S \subsetneq X \cup \{x\}$, then $i(S)$ is an open set in X. If $i(S) \neq \varnothing$, then $b_1 i(S) = X$, so $bi(S) = b_1 i(S) \cup \{x\} = X \cup \{x\}$, whence $ibi(S) = i(X \cup \{x\}) = X \cup \{x\} = bi(S)$. If $i(S) = \varnothing$, then $ibi(S) = ib(\varnothing) = \varnothing = b(\varnothing) = bi(S)$. Hence $ibi = bi$, that is $(X \cup \{x\}, \mathcal{T}^x)$ is extremally disconnected.

Conversely, if X has disjoint non-empty open sets A, B, then $i(A) = i_1(A) = A \neq \varnothing$ so $x \in bi(A)$. Also, $bi(A) = b(A) = b_1(A) \cup \{x\} \subsetneq X \cup \{x\}$ (as $b_1(A) \subseteq X \backslash B$ and hence $b_1(A) \neq X$). Because $x \notin ibi(A)$ we conclude that $ibi \neq bi$ and so $(X \cup \{x\}, \mathcal{T}^x)$ is not extremally disconnected. Hence $(X \cup \{x\}, \mathcal{T}^x)$ *is extremally disconnected if and only if X has no disjoint non-empty open sets.*

(2) If $(X, \mathcal{T})$ is an OU space, let S be dense in $(X \cup \{x\}, \mathcal{T}^x)$ (so that $b_1(S \backslash \{x\}) = X$ also). If $x \in i(S)$ then trivially $i(S) = X \cup \{x\} = S$ is dense. If $x \notin i(S)$, then $i(S) = i(S \backslash \{x\}) = i_1(S \backslash \{x\})$, which is dense in the OU-space $(X, \mathcal{T})$. Therefore $bi(S) = b(i_1(S \backslash \{x\})) = b_1 i_1(S \backslash \{x\}) \cup \{x\} = X \cup \{x\}$ as well. That is, $(X \cup \{x\}, \mathcal{T}^x)$ is an OU-space.

Conversely, if $(X \cup \{x\}, \mathcal{T}^x)$ is OU and T is dense in $(X, \mathcal{T})$ then as above, T is dense in $(X \cup \{x\}, \mathcal{T}^x)$ so $i_1(T) = i(T)$ is dense in $(X \cup \{x\}, \mathcal{T}^x)$ and hence in $(X, \mathcal{T})$. Thus $(X \cup \{x\}, \mathcal{T}^x)$ *is OU if and only if $(X, \mathcal{T})$ is OU.*

(3) No matter what $(X, \mathcal{T})$ is like, the set $\{x\}$ is closed but not open in $(X \cup \{x\}, \mathcal{T}^x)$ so this space cannot be a partition space, so $(X \cup \{x\}, \mathcal{T}^x)$ *has K-number greater than* 6.

All parts of the proposition follow easily from (1), (2) and (3). □

This proposition allows for the easy construction of spaces with a minimal number of points for each of the given K-numbers. There can be no three point Kuratowski space, since a Kuratowski space cannot have k-number 4 (by Lemma 2.8) and the three non-homeomorphic spaces on three points with k-number greater than 4 (as constructed in [**1**]) are either extremally disconnected or OU. To make a four point Kuratowski space, start with the partition space $X := \{x, y, z\}$ and $\mathcal{T} := \{\varnothing, \{x\}, \{y, z\}, X\}$. By Proposition 3.11, $(X \cup \{w\}, \mathcal{T}^w)$ is a Kuratowski space. In fact this example is homeomorphic to the example of a non-extremally disconnected and non-OU space presented in [**10**]. A two point space $(\{x, y\}, \mathcal{T})$ is either anti-discrete, discrete or contains one closed singleton and one open singleton (extremally disconnected and OU). By Proposition 3.11, $(\{x, y, z\}, \mathcal{T}^z)$ is extremally disconnected but not OU if $(\{x, y\}, \mathcal{T})$ is anti-discrete, and OU but not extremally disconnected if $(\{x, y\}, \mathcal{T})$ is discrete. This last example is a door space (k-number 4) with K-number 10.

3.3. Continuous images. In a similar way to Theorem 3.7, we have the following obvious corollary of Proposition 3.4 and Lemma 3.5.

Theorem 3.12. *If* $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ *is a continuous open surjection then* $K(Y, \mathcal{U}) \le K(X, \mathcal{T})$.

Again the situation for general continuous images differs enormously—every space is a continuous image of a discrete space (with K-number equal to 2). Even amongst the images of a closed continuous map we may lose any existing identifications between Kuratowski operators.

Proposition 3.13. *Every Boolean space is the image under a continuous closed map from an extremally disconnected Boolean space.*

Proof. It is well known that every Boolean algebra embeds into a complete Boolean algebra—namely the algebra of all subsets of some set. Let X be an arbitrary Boolean space. Now let B be a complete Boolean algebra embedding X^*. Then under the Stone duality, there is a continuous surjection from B^* onto $(X^*)^*$, which is homeomorphic to X. This is also a closed map, because continuous maps between compact Hausdorff spaces are closed (Proposition 3.6 (ii)), however B^* is extremally disconnected because B was complete (Proposition 3.6 (i)). □

The usual Cantor space is a full Kuratowski space and also a Boolean space. Proposition 3.13 shows that it is the image under a continuous closed map from an extremally disconnected Boolean space (the Stone-Čech compactification of a countably infinite discrete space for example).

3.4. Sums. Subsection 3.1 suggests that topological sums have a reasonably natural place in the study of Kuratowski operators. To make this more explicit, we define the notion of a *variety of Kuratowski-acts* to be an equationally defined class of Kuratowski-acts. Typical examples are provided by the possible topological identifications described in Subsection 2.1. Thus the variety satisfying $ibi \approx bi$ and $bib \approx ib$ is the class of Kuratowski-acts corresponding to extremally disconnected spaces.

As with equationally defined classes of algebraic structures—groups and rings for example—varieties of Kuratowski-acts are closed under taking isomorphic copies of sub-acts, homomorphic images and direct products. Proposition 3.4 now gives the following corollary.

Corollary 3.14. *Let $\mathcal{V}$ be a variety of Kuratowski acts.*

(i) *If $(X, \mathcal{T}) = \sum_i (X_i, \mathcal{T}_i)$ and each $\Phi(X_i, \mathcal{T}_i) \in \mathcal{V}$, then $\Phi(X, \mathcal{T}) \in \mathcal{V}$.*
(ii) *If $(X, \mathcal{T})$ is a space with $\Phi(X, \mathcal{T}) \in \mathcal{V}$ and $W \subseteq X$ is open, then $\Phi(W, \mathcal{T}_W) \in \mathcal{V}$.*
(iii) *If $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ is a surjective continuous open map, and $\Phi(X, \mathcal{T}) \in \mathcal{V}$ then $\Phi(Y, \mathcal{U}) \in \mathcal{V}$.*

Parts (ii) and (iii) are effectively just Theorems 3.7 and 3.12 respectively, inasmuch as each of the diagrams of Figure 2.1 defines a variety of Kuratowski acts. The following lemma on sum spaces should be clear.

Lemma 3.15. *Let $\{(X_i, \mathcal{T}_i) : i \in I\}$ be a family of spaces with the X_i pairwise disjoint and let o_1 and o_2 be two Kuratowski operators. Then o_1 and o_2 are equal on the sum space $\sum_{i\in I} X_i$ if and only if they are equal on each $(X_i, \mathcal{T}_i)$. Furthermore if $S := \dot\cup_{i\in I} S_i$ is a subset of $\dot\cup_{i\in I} X_i$ then o_1 and o_2 agree on S if and only if they agree on each S_i.*

While the sum space construction preserves K-numbers as much as possible, it turns out to be a convenient construction for increasing k-numbers. In the following we use the notation $X \times \{1, \ldots, n\}$ to denote the disjoint union of n copies of a set X.

Proposition 3.16. *Let $(X, \mathcal{T})$ have Kuratowski monoid* **K**.

(i) *If $K((X, \mathcal{T})) = 2$ then $(X, \mathcal{T})$ is a full space.*
(ii) *If $K((X, \mathcal{T})) = 6$ then the sum space on $X \times \{1, 2\}$ is a full space with Kuratowski monoid* **K**.
(iii) *If $K((X, \mathcal{T})) = 8$ then the sum space on $X \times \{1, 2\}$ is a full space with Kuratowski monoid* **K**.
(iv) *If $K((X, \mathcal{T})) = 10$ then the sum space on $X \times \{1, 2, 3\}$ is a full space with Kuratowski monoid* **K**.
(v) *If $K((X, \mathcal{T})) = 14$ then the sum space on $X \times \{1, 2, 3, 4\}$ is a full space with Kuratowski monoid* **K**.

Proof. Lemma 3.15 implies that the Kuratowski monoid of the sum space on copies of X is equal to that of $(X, \mathcal{T})$. Case (i) is trivial. For (ii) note that $X \times \{1, 2\}$ is a partition space but not discrete (since $(X, \mathcal{T})$ is not discrete) or anti-discrete and so has k-number 6. For (iii) there are three possibilities. If the space has k-number 6 or 8 then Lemma 3.15 implies that the topology on $X \times \{1, 2\}$ is full. If the space has k-number 4 then by Lemma 2.8, $(X, \mathcal{T})$ is either a door space or $\mathcal{T}\backslash\{\varnothing\}$ is a filter. However if $\mathcal{T}\backslash\{\varnothing\}$ is a filter then every neutral set is dense and co-dense. However $K((X, \mathcal{T})) = 8$ implies that $(X, \mathcal{T})$ is an extremally disconnected OU-space, whence every dense set has dense interior and so it follows that there are no neutral sets. That is, $(X, \mathcal{T})$ is a door space. Now there must be a non-clopen set A in X if $(X, \mathcal{T})$ has K-number 8. Table 2.1 shows that there are only two (complementary) possibilities for such a set in a door space and the corresponding equalities between Kuratowski operators and Lemma 3.15 imply that k-number of $A \times \{1\} \cup (X\backslash A) \times \{2\}$ in the sum topology is exactly 8. Hence the sum space is full.

For case (iv) there are two possibilities: the space could be OU or extremally disconnected. The result will follow from Lemma 3.15 in the case where $k((X,\mathcal{T})) \geq 6$. If $k((X,\mathcal{T})) = 4$ then by Lemma 2.8 $(X,\mathcal{T})$ is a door space or $\mathcal{T}\backslash\{\varnothing\}$ is a filter.

Subcase 1. $(X,\mathcal{T})$ is a door space.
Then $(X,\mathcal{T})$ is an OU space by a result in [**2**]. Since the space cannot also be extremally disconnected (the K-number is 10), Table 2.1 implies that there is a set A with for which $A = ibi(A) = i(A) = ib(A)$ and $b(A) = bib(A) = bi(A)$ and for which no other equalities hold. Since $i \neq ib$ in an OU space there must also be a set B such that $i(B) \neq ib(B)$; in fact it can be seen from Table 2.1 that for B we have $B = i(B)$ and $ib(B) = ibi(B) = bi(B) = bib(B)$. Then in the sum space the set $A \times \{1\} \cup B \times \{2\} \cup (X\backslash B) \times \{3\}$ must have the operators $\mathsf{id}, i, b, ib, bi$ acting differently. Hence the sum space has k-number 10 as required. An example where the sum on only two copies of X will not suffice to provide a full space is the door space on $\{x,y,z\}$ with basis $\{\varnothing, \{x\}, \{y\}, \{x,y,z\}\}$.

Subcase 2. $\mathcal{T}\backslash\{\varnothing\}$ is a filter in $(X,\mathcal{T})$ but $(X,\mathcal{T})$ is not a door space.
Because the K-number is 10, there must be a non-closed open set A and a neutral set B. Since there is no non-empty open set disjoint from A, we have $b(A) = X$, while B is also dense but with empty interior. Then $A \times \{1\} \cup (X\backslash A) \times \{2\} \cup B \times \{3\}$ has k-number 10 as required.

For part (v), the only difficult case is when the k-number of $(X,\mathcal{T})$ is less than or equal to 6. Lemma 2.8 implies that spaces with k-number 4 are either OU spaces (the door space case) or extremally disconnected (the case when $\mathcal{T}\backslash\{\varnothing\}$ is a filter). Hence it remains to consider the case when $k((X,\mathcal{T})) = 6$.

There must be a subset $A \subseteq X$ with k-number at most 6 for which $ib(A) \neq ibi(A)$; Table 2.1 implies that there is only one possibility and that we must also have $bi(A) \neq bib(A)$. There must also be a subset $B \subseteq X$ with k-number at most 6 for which $ib(B) \neq bib(B)$; Table 2.1 shows that every possibility also has $bi(B) \neq ibi(B)$. Then in the sum space on $X \times \{1\} \cup X \times \{2\}$, the set $A \cup B$ must have at most the following equalities: $i = ibi$, $b = bib$. There is a subset $C \subseteq X$ such that $i(C) \neq ibi(C)$ showing that in the sum space on $X \times \{3\} \cup X \times \{4\}$, the operators i, b, bib, ibi act differently on the set $C \times \{3\} \cup (X\backslash C) \times \{4\}$. Hence on the sum space on $X \times \{1,2,3,4\}$, the set $A \times \{1\} \cup B \times \{2\} \cup C \times \{3\} \cup (X\backslash C) \times \{4\}$ is a 14-set. A Kuratowski space where the sum of only two copies of X will not suffice to produce a 14-set is the four point Kuratowski space on $\{1,2,3,4\}$ with basis $\{\varnothing, \{1\}, \{2,3\}, \{1,2,3,4\}\}$. We do not know of a space where more than three copies are required, or in fact any Kuratowski space with k-number 6.

This completes the proof. □

3.5. Products. To begin with we note that the number of distinct Kuratowski operators on a product space is at least that on the individual spaces.

Lemma 3.17. *Let $\{(X_\lambda, \mathcal{T}_\lambda) : \lambda \in \Lambda\}$ be a family of topological spaces and $\Gamma \subseteq \Lambda$. If $\Pi_{\lambda\in\Lambda} X_\lambda$ and $\Pi_{\gamma\in\Gamma} X_\gamma$ are given the product topologies, then the Kuratowski monoid of $\Pi_{\gamma\in\Gamma} X_\gamma$ is a quotient of that of $\Pi_{\lambda\in\Lambda} X_\lambda$.*

Proof. It is well known and easy to prove that the projection map π from $\Pi_{\lambda\in\Lambda} X_\lambda$ onto $\Pi_{\gamma\in\Gamma} X_\gamma$ is open and continuous map (see [**27**, Theorem 3.2]). Proposition 3.4 and Lemma 3.5 now show that the Kuratowski monoid of $\Pi_{\gamma\in\Gamma} X_\gamma$ is a quotient of that of $\Pi_{\lambda\in\Lambda} X_\lambda$. □

This lemma immediately implies the following result.

Lemma 3.18. *Let $\{(X_\lambda, \mathcal{T}_\lambda) : \lambda \in \Lambda\}$ be a family of spaces and I, J be disjoint subsets of Λ. If* **U** *and* **V** *are the Kuratowski monoids of $\Pi_{i\in I} X_i$ and $\Pi_{j\in J} X_j$ (resp.) given the product topology, then the product topology on $\Pi_{\lambda\in\Lambda} X_\lambda$ has a Kuratowski monoid that is an upper bound of* **U** *and* **V** *in the diagram of Figure 2.3. In particular, $K(\Pi_{\lambda\in\Lambda} X_\lambda) \geq \max\{K(\Pi_{i\in I} X_i), K(\Pi_{j\in J} X_j)\}$.*

This shows, for example that the class of Kuratowski spaces is closed under taking arbitrary products and that the product of an extremally disconnected but not OU space (such as a non-degenerate anti-discrete space) with an OU but not extremally disconnected space is a Kuratowski space.

We now consider the stability of Kuratowski operators on finite products. Clearly the product of two discrete spaces is again a discrete space. Likewise, the product of two partition spaces is again a partition space; to prove this we first make the following observation.

Lemma 3.19. *If $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are spaces for which $\mathcal{T}_X$ and $\mathcal{T}_Y$ are closed under arbitrary intersections, then the topology on the product space is also closed under arbitrary intersections.*

Proof. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be as in the statement of the lemma and let $\{A_i \times B_i : i \in I\}$ be open basis sets in the product space (with the standard basis). Then $\cap_{i\in I} A_i \times B_i = (\cap_{i\in I} A_i) \times (\cap_{i\in I} B_i)$ which is a basic open set. The claim now follows because of distributivity of arbitrary intersections over arbitrary unions. □

A Hausdorff space whose topology is closed under arbitrary intersections is discrete (because every set is a union of its closed subsets). However the topology on a partition space and on any finite space is closed under arbitrary intersections. Further examples of interest in algebra and computer science arise from any partially ordered set by taking the so-called *Alexandrov topology* consisting of all downsets (see [**48**, p. 22], for example).

Lemma 3.20. *If $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are partition spaces then so is the product space $(X \times Y, \mathcal{T}_{X\times Y})$.*

Proof. First note that $\mathcal{T}_X$ and $\mathcal{T}_Y$ are closed under arbitrary intersections and so by Lemma 3.19, $\mathcal{T}_{X\times Y}$ is closed under arbitrary intersections. Now if $S := \cup_{i\in I} A_i \times B_i$ is an open set, then the complement of S in $X \times Y$ is equal to $\cap_{i\in I}\big(((X\backslash A_i) \times Y) \cup (X \times (Y\backslash B_i))\big)$. Since each A_i and B_i is clopen, this is an intersection of open sets and hence S has open complement and therefore is clopen. □

The corresponding result for the class of all extremally disconnected spaces does not hold. An interesting counterexample showing this is the single ultrafilter topology, which may be described as follows. Let U be a non-principal ultrafilter on a countably infinite discrete space X and let $X \cup \{U\}$ be given the topology consisting of the open sets of X along with $\{A \cup \{U\} : A \in U\}$. As is shown in [**44**, Example 114], this is extremally disconnected. In fact since there is only one non-isolated point it is also a door space and hence OU. In the product topology on $(X \cup \{U\}) \times (X \cup \{U\})$, the set $S := \{(x, x) : x \in X\}$ is open and has closure $S \cup \{(U, U)\}$ which in turn has interior equal to S. Hence the closure of an open set need not be clopen and the space is not extremally disconnected. (For another example, see [**18**, Exercise 6N].)

In the positive direction however, we have the following.

Lemma 3.21. *If* $(X, \mathcal{T}_X)$ *and* $(Y, \mathcal{T}_Y)$ *are extremally disconnected spaces with* $\mathcal{T}_X$ *and* $\mathcal{T}_Y$ *closed under arbitrary intersections, then the product space* $(X \times Y, \mathcal{T}_{X\times Y})$ *is extremally disconnected.*

Proof. First note that by Lemma 3.19, any union of closed sets in the product space is again closed. Now let S be an open set in the product topology. We need to show that the closure of S is open. Now $S = \cup_{i\in I} A_i \times B_i$ for some open basis sets $A_i \times B_i$. Then $b(\cup_{i\in I} A_i \times B_i) \subseteq \cup_{i\in I} b(A_i \times B_i)$, since by Lemma 3.19, the right hand side is a closed set containing S. However the reverse containment holds in any space and hence $b(\cup_{i\in I} A_i \times B_i) = \cup_{i\in I} b(A_i \times B_i)$. Now $b(S) = b(\cup_{i\in I} A_i \times B_i) = \cup_{i\in I} b(A_i \times B_i) = \cup_{i\in I} b(A_i) \times b(B_i)$ which is a union of open sets as required. □

The following example is given by Aull [**2**] as an example of a T_1 door space and hence is OU. Let X be an infinite set and give it the topology consisting of the empty set with all members of some non-principal ultrafilter U. In the product topology on $X \times X$, the set $\{(x, x) : x \in X\}$ contains no open subsets because X contains no isolated points. That is, $\{(x, x) : x \in X\}$ is co-dense. Now a basic open set is of the form $A \times B$ for $A, B \in U$ and so $A \cap B \neq \varnothing$. Then for each, $x \in A \cap B$ we have $(x, x) \in A \times B$ and hence $X \backslash \{(x, x) : x \in X\}$ contains no non-empty open sets, showing that $\{(x, x) : x \in X\}$ is both dense and co-dense in $X \times X$. This shows that $X \times X$ is not an OU space. Note however that the product space contains no disjoint non-empty sets and so is extremally disconnected.

Despite this example, we can prove that a quite large class of OU-spaces is closed under finite products. We first note the following intermediate lemma.

Lemma 3.22. *A space* $(X, \mathcal{T})$ *is OU provided every non-empty open set contains an isolated point. For finite spaces this condition is if and only if.*

Proof. Let $(W, \mathcal{U})$ be a space in which every non-empty open set contains an isolated point. If D is dense, then for every non-empty open set G there is an isolated point $x_G \in G$ and $D \cap \{x_G\} \neq \varnothing$. That is, $x_G \in D$ and so $x_G \in i(D)$. But then $i(D) \cap G \neq \varnothing$ for all G, so $i(D)$ is dense and $(W, \mathcal{U})$ is an OU space.

To prove the second statement, we first show that every OU space has the following property: every non-empty open set is either an isolated point or contains a proper and non-empty open subset. Suppose there is a space $(W, \mathcal{U})$ with an open set A such that $|A| > 1$ but A has no proper open subsets. Then the open subspace on A is resolvable (see Definition 2.2 (ii)), contradicting Theorem 2.3 and Definition 2.2 (iii).

Now let $(W, \mathcal{U})$ be a finite OU-space and let A be open and non-empty in $\mathcal{U}$. If A is a singleton we are done. Otherwise there is a non-empty open subset B with $B \subsetneq A$. When W is finite, this implies that A must contain an isolated point. □

Lemma 3.23. *Let* $(X, \mathcal{T}_X)$ *and* $(Y, \mathcal{T}_Y)$ *be OU spaces with the property that every non-empty open set contains an isolated point. Then the product space* $(X \times Y, \mathcal{T}_{X\times Y})$ *is an OU space.*

Proof. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be OU spaces for which every non-empty set contains an isolated point. If S is open and non-empty in $X \times Y$, then there are non-empty sets $A \in \mathcal{T}_X$ and $B \in \mathcal{T}_Y$ such that $A \times B \subseteq S$. If $u \in A$ and $v \in B$ are isolated, that is $\{u\} \in \mathcal{T}_X$ and $\{v\} \in \mathcal{T}_Y$, then $\{(u, v)\} = \{u\} \times \{v\}$ is open in $X \times Y$. That is, (u, v) is an isolated point in S. Thus $X \times Y$ is an OU space by Lemma 3.22. □

Spaces in which every non-empty open set contains an isolated point appear to be rather common amongst the known examples of OU-spaces. Aside from finite examples, witness the spaces related to the Arens-Fort space [**44**, Examples 23–27], the Post Office metric space [**44**, Example 139] and the single ultrafilter topology. More generally, any Hausdorff door space contains at most one non-isolated point [**27**, p. 76] and therefore has this property. Lemma 3.23 shows that any finite product of such spaces is an OU space.

For infinite products the situation is substantially less stable. For example it is well known that the product topology on the denumerable power of the two element discrete space is homeomorphic to the usual topology on the Cantor set, which is a Kuratowski space by Proposition 2.11. In fact we have the following.

Theorem 3.24. *If $\{(X_i, \mathcal{T}_i) : i \in I\}$ is an infinite family of non-connected spaces then the product space on $\prod_{i \in I} X_i$ is a Kuratowski space.*

Proof. For each X_i let G_i and H_i be disjoint non-empty open sets for which $G_i \cup H_i = X_i$. Let $\{0, 1\}$ be given the discrete topology and define a map $f_i : X_i \to \{0, 1\}$ by setting

$$f_i(x) = \begin{cases} 1 & \text{if } x \in G_i \\ 0 & \text{if } x \in H_i. \end{cases}$$

This is routinely seen to be continuous and since the topology on $\{0, 1\}$ is discrete, f_i is an open map.

We now consider $\Pi_{i \in I} X_i$ given the product topology and let $f : \Pi_{i \in I} X_i \to \Pi_{i \in I}\{0, 1\}$ be defined by $f((x_i)_I) = (f_i(x_i))_I$. We now show that f is a continuous open map. Theorem 3.12 then implies that the Kuratowski monoid of $\Pi_{i \in I} X_i$ is at least as large as that of the product of I copies of the two element discrete space. By Lemma 3.17 this in turn is at least as large as the Kuratowski monoid of a countable product of two element discrete spaces, which by Proposition 2.11 has K-number 14. This will show that $K(\Pi_{i \in I} X_i) = 14$, as required.

First, for $i \in I$, let A_i be open in $\{0, 1\}$ with almost all A_i equal to $\{0, 1\}$. Then

$$\begin{aligned} f^{-1}(\Pi_{i \in I} A_i) &= \{(x_i)_I : \ f((x_i)_I) \in \Pi_{i \in I} A_i\} \\ &= \{(x_i)_I : (\forall i \in I) \ f_i(x_i) \in A_i\} \\ &= \{(x_i)_I : (\forall i \in I) \ x_i \in f_i^{-1}(A_i)\} \\ &= \Pi_{i \in I} f_i^{-1}(A_i) \end{aligned}$$

which is a basic open set in $\Pi_{i \in I} X_i$ since each $f_i^{-1}(A_i)$ is open in X_i (it is either G_i, H_i or X_i) and almost all are equal to X_i. Therefore f is continuous for the product topologies.

Now let B_i be open in X_i, with B_i equal to X_i for almost all $i \in I$. Then

$$\begin{aligned} f(\Pi_{i\in I}B_i) &= \{f((b_i)_I) : (b_i)_I \in \Pi_{i\in I}B_i\} \\ &= \{(f_i(b_i))_I : (b_i)_I \in \Pi_{i\in I}B_i\} \\ &= \Pi_{i\in I}f_i(B_i). \end{aligned}$$

For almost all i, $f(B_i) = \{0,1\}$ and therefore $f(\Pi_{i\in I}B_i)$ is open in $\Pi_{i\in I}\{0,1\}$. Therefore f takes basic open sets in $\Pi_{i\in I}X_i$ to basic open sets in $\Pi_{i\in I}\{0,1\}$. Now let $U = \cup F_\lambda$ be an arbitrary open set in $\Pi_{i\in I}X_i$, where the F_λ are basic open sets. Then $f(U) = f(\cup F_\lambda) = \cup f(F_\lambda)$ which is a union of basic open sets in $\Pi_{i\in I}\{0,1\}$ and therefore open. Hence f is an open map as required. □

The following theorem is trivial.

Theorem 3.25. *If $\{(X_i, \mathcal{T}_i) : i \in I\}$ is a family of spaces each containing no disjoint open sets then the product topology on $\prod_{i\in I} X_i$ also has no disjoint open sets and so is extremally disconnected.*

In fact this is about the only situation in which an infinite product can fail to be a Kuratowski space.

Theorem 3.26. *Let $\{(X_\lambda, \mathcal{T}_\lambda) : \lambda \in \Lambda\}$ be an infinite family of spaces with $|X_\lambda| > 1$. If the product topology on ΠX_λ is not a Kuratowski space then each space $(X_\lambda, \mathcal{T}_\lambda)$ is extremally disconnected and there is a cofinite subset Γ of Λ, such that for each $\gamma \in \Gamma$ the space $(X_\gamma, \mathcal{T}_\gamma)$ has no disjoint non-empty open sets.*

Proof. We first show that an infinite product of non-singleton spaces is not an OU-space. Let a_λ and b_λ be distinct points in X_λ and consider the subset

$$U := \{(x_\lambda)_\Lambda : x_\lambda \neq a_\lambda \text{ for only finitely many } \lambda \in \Lambda\}.$$

We show that this set is both dense and co-dense in the product topology. Let $A = \Pi A_\lambda$ be a basic open set in ΠX_λ. Now $\Lambda_A := \{\gamma : A_\lambda = X_\lambda\}$ is co-finite and so there are points $(x_\lambda)_\Lambda$ and $(y_\lambda)_\Lambda$ in A such that for $\gamma \in \Lambda_A$, $x_\gamma = b_\gamma$ and $y_\gamma = a_\gamma$. Then $(x_\lambda)_\Lambda \notin U$, while $(y_\lambda)_\Lambda \in U$. Hence both U and its complement are dense in the product topology. By Theorem 2.3, ΠX_λ is not an OU-space under the product topology.

This means that if the product fails to be a Kuratowski space, it is extremally disconnected, and then by Lemma 3.18, each $(X_\lambda, \mathcal{T}_\lambda)$ is extremally disconnected. Let us now assume that each $(X_\lambda, \mathcal{T}_\lambda)$ is extremally disconnected.

Say there is an infinite subset $\Xi \subseteq \Lambda$ such that for $\xi \in \Xi$, $(X_\xi, \mathcal{T}_\xi)$ contains disjoint non-empty open sets, $A_\xi, B_\xi \subseteq X_\xi$. Now $b(A_\xi)$ is clopen and disjoint from B_ξ and so each $(X_\xi, \mathcal{T}_\xi)$ is non-connected. Therefore Theorem 3.24 implies that the product topology on $\Pi_{\xi\in\Xi}X_\xi$ is a Kuratowski space. Then Lemma 3.18 shows that the product topology on $\Pi_{\lambda\in\Lambda}X_\lambda$ is a Kuratowski space. □

In view of Theorem 3.25, for single spaces this theorem can be written as follows.

Corollary 3.27. *Let $(X, \mathcal{T})$ be a non-degenerate space. If Λ is an infinite set then the product topology on X^Λ is a Kuratowski space if and only if X contains two disjoint non-empty open subsets.*

4. Variants of The Complement-closure Problem

Several natural variants of the complement-closure problem for sets and spaces seem quite natural. In this section we examine many of these variants, surveying the existing solutions in the literature and solving the obvious remnants. The question as to the number of sets obtainable from a given subset of a topological space using some set $\mathcal{O}$ of operators will be called the *Kuratowski $\mathcal{O}$-problem.* An operator o derived by composing members of some set $\mathcal{O}$ of operators will be called an *$\mathcal{O}$-operator.*

The reader is also directed to the recent article [**38**] by David Sherman, which has some overlap with this section and provides an excellent introduction to the Kuratowski Closure-Complement Theorem.

4.1. Topological variants.

4.1. Topological variants. In this section we will primarily look at various subsets of the operators $\{a, b, i, \vee, \wedge\}$ where $\vee$ (*join*) and $\wedge$ (*meet*) are the *binary operators* corresponding to union and intersection. More specifically, by a binary operator $*$ on the set of subsets of a set X we mean a function $* : 2^X \times 2^X \to 2^X$. For example the operator $ab \wedge a$ applied to a set A gives $ab(A) \cap a(A)$. Evidently, as both $\wedge$ and $\vee$ are simply modelling the notions of intersection and union of subsets of a set, they will satisfy the usual properties of a distributive lattice; namely they are associative, commutative and idempotent, either one distributes over the other and $x \wedge (x \vee y) = x = x \vee (x \wedge y)$ hold for any x and y.

It is of interest to recall a remarkable result established by McKinsey and Tarski (a combination of Theorems 3.8, 4.15 and 5.10 in [**32**]).

Theorem 4.1. [**32**] *Let $\mathcal{O} = \{a, b, i, \wedge, \vee\}$ and let u and v be $\mathcal{O}$-operators and let X denote any of the following spaces with their usual topologies: the usual Cantor space; $\mathbb{R}^n$ (for any $n \in \mathbb{N}$); $\mathbb{Q}$. Then u and v are identical on every topological space if and only if u and v are identical on every finite topological space and if and only if u and v are identical on X. Furthermore, there is an algorithm that verifies when given any two such $\mathcal{O}$-operators u and v, whether or not u and v coincide on X.*

Let $\mathcal{O}$ be a set of $\{a, b, i, \wedge, \vee\}$-operators. If the Kuratowski $\mathcal{O}$-problem has a finite solution, say n, then it follows from Theorem 4.1, that there is a subset of the reals and a subset of a finite topological space on which each of the n possible $\mathcal{O}$-operators act differently. If infinitely many sets are obtainable from a subset of a topological space using the operators in $\mathcal{O}$, then there is a subset of the reals for which the infinitely many distinct $\mathcal{O}$-operators are all different.

The approach taken in [**32**] is from an abstract algebraic perspective. McKinsey and Tarski make use of closure algebras: Boolean algebras endowed with a unary operation C satisfying $x \leq C(x)$, $C(x \vee y) = C(x) \vee C(y)$, $C(C(x)) = C(x)$, $C(0) = 0$. An example is the Boolean algebra of all subsets of a topological space $(X, \mathcal{T})$ with C the usual closure operator. The sum space construction of Section 3.4 is easily seen to correspond to direct products of the corresponding closure algebras. Closure algebras have also been of interest in the study of the S4 modal logics [**6**] and indeed some of the collapses in Theorem 2.1 correspond to important classes of modal logics. For example, the system S5 corresponds in a natural way to partition spaces (see [**24**, Chapter 3]).

The Kuratowski $\{a, b, \vee\}$- and $\{a, b, \wedge\}$-problem.
This problem is solved by Kuratowski in his original paper (see page 197 of [**29**]): there is no bound on the number of sets. The result is also presented in [**32**] in order to show that the free closure algebra on one generator is infinite, however in [**5**] (a paper concerning the same class of structures), the following elementary example is suggested.

Example 4.2. *Let $\mathcal{T}$ denote $\{\{1, 2, 3, \ldots, n\} : n \in \mathbb{N}\} \cup \{\varnothing, \mathbb{N}\}$ and let $\mathbf{B}$ denote the topological space $(\mathbb{N}, \mathcal{T})$. Then infinitely many subsets may be obtained from the subset $\{1, 3, 5, \ldots\}$ by taking closures, complements and intersections.*

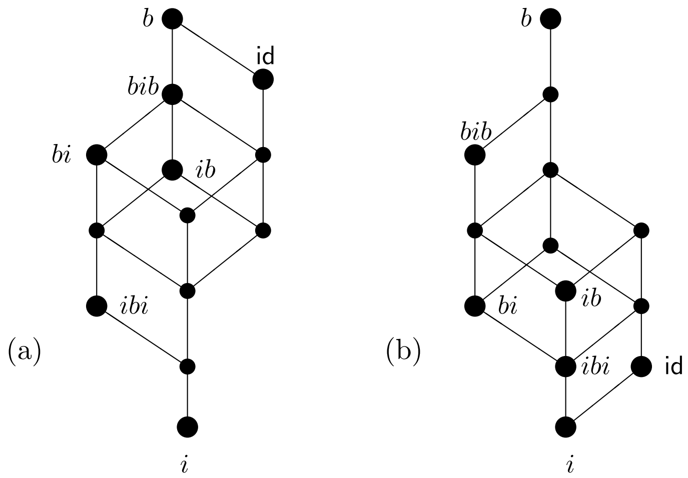
One does not need the full power of the operators amongst $\{a, b, \vee\}$ to produce infinitely many sets from a single set in $\mathbf{B}$. As an example we consider the derived operators of set subtraction ($X \backslash Y := X \cap a(Y)$) and of implication ($X \rightarrow Y := Y \cap a(X)$). Note that $X \backslash Y$ is just the complement of $X \rightarrow Y$. Inductively define operators $s_0, s_1, s_2, \ldots$ and $t_0, t_1, t_2, \ldots$ by setting $s_0, t_0 := \mathsf{id}$, $s_{k+1} := b(s_k) \backslash s_k$, and $t_{k+1} := t_k \rightarrow i(t_k)$. The reader will verify that each of the operators $s_0, s_1, \ldots$ produce distinct sets when applied to the subset $\{2, 4, 6, \ldots\}$ in $\mathbf{B}$, while the operators $t_0, t_1, \ldots$ produce distinct sets when applied to the subset $\{1, 3, 5, \ldots\}$ in $\mathbf{B}$.

The Kuratowski $\{b, i, \vee\}$ problem.
This problem is the topic of American Mathematical Monthly Problem 5996 [**41**] (a solution is given in [**49**]): at most 13 sets can be obtained from using closure, intersection and union from a subset of a topological space. (This result is also rediscovered in [**38**].) The Kuratowski $\{b, i, \wedge\}$ problem follows dually since if A is subset of a topological space from which 13 sets can be obtained by taking closures, interiors and unions, then the complement of A yields the same number of sets by taking closures, interiors and intersections. For arguments below we in fact prefer the $\{b, i, \wedge\}$ formulation of this problem and will discuss this only.

The 13 distinct $\{b, i, \wedge\}$-operators admit a natural partial order which is in fact merely the meet semilattice freely generated by the partially ordered set depicted in Figure 1.1. By this we mean it is the largest meet semilattice generated by the elements of the set in Figure 1.1 subject to the order that is already present on these operators. This object can be constructed by the introduction of all possible 'formal meets' between existing incomparable elements of the seven element partially ordered set in Figure 1.1. For example, it is conceivable that the meet of bi with ib is not the operator ibi, since the diagram in Figure 1.1 only shows that $ibi \leq bi$ and $ibi \leq ib$. Thus we adjoin a new element denoted by $bi \wedge ib$. On the other hand, the meet of the elements ib and ibi must in fact be ibi itself, since $ibi \leq ib$. The difficult part of the proof in [**49**] is showing that this enlarged set of operators is closed under taking of closures and interiors.

The ordered set of 13 $\{b, i, \wedge\}$-operators is given in Figure 4.1 (a); diagram (b) is the $\{b, i, \vee\}$ dual (here we have used larger points to indicate the even Kuratowski operators). For example, the left diagram shows that $b(bi \wedge ib) = bi$, because bi is the smallest closed operator above $bi \wedge ib$ in the diagram.

In [**33**] it is shown that the smallest space containing a subset on which all 13 operators in Figure 4.1 (a) act differently has 9 points. Analogously to [**1**], the



(a) (b)

FIGURE 4.1. The 13 different operators produced by b, i and $\wedge$ and by b, i and $\vee$.

maximum possible number of distinct operators of this kind is also given for each space with fewer than 9 points (the sequence of possible maximums starting from a 1-point space is $1, 3, 3, 5, 7, 8, 10, 12, 13$).

The Kuratowski $\{b, i, \vee, \wedge\}$ problem.
If A is a given subset of a topological space, and B is a set of obtained from A by way of iteratively applying a collection, $K \subseteq \{a, b, i, \vee, \wedge\}$ of operators, then we say that B is a K-set of A. The Kuratowski $\{b, i, \vee, \wedge\}$ problem does not seem to have been investigated in the literature, but the solution we give here was independently discovered by Sherman [**38**].

The operations of union and intersection distribute over each other, and hence taking all unions and intersections of the at most 7 $\{b, i\}$-sets of a given set A we obtain a (finite) distributive lattice (ordered by set inclusion). Denote this lattice by L. Note that closures distribute over finite unions and that every element of L can be written as a finite union of intersections of $\{b, i\}$-sets of A. So the closure of an element of L is equal to a union of a finite family of closures of intersections of $\{b, i\}$-sets of A. But the solution to the Kuratowski $\{b, i, \wedge\}$-problem indicates that each member of this family is itself equal to an intersection of $\{b, i\}$-sets of A. Hence L is closed under the taking of closures.

A dual argument applies for the taking of interiors. To demonstrate the idea, consider the following reduction (the third equality uses the solution to the $\{b, i, \vee\}$-problem):

$$\begin{aligned} i\big(ib(A) \cup (bi(A) \cap A)\big) &= i\big((ib(A) \cup bi(A)) \cap (ib(A) \cup A)\big) \text{ (by distributivity)} \\ &= i\big(ib(A) \cup bi(A)\big) \cap i\big(ib(A) \cup A\big) \\ &= ib(A) \cap ib(A) \\ &= ib(A). \end{aligned}$$

Therefore, on any space, the operator $i(ib \vee (bi \wedge \mathsf{id}))$ coincides with ib.

All this goes to tell us that to find an upper bound for a solution to the Kuratowski $\{b, i, \vee, \wedge\}$-problem we need only construct all new formal joins in the meet semilattice of Figure 4.1 (a). The resulting 35 possible operators and their associated lattice Hasse diagram are depicted in Figure 4.2 (here larger points indicate operators in common with Figure 4.1 (a)).

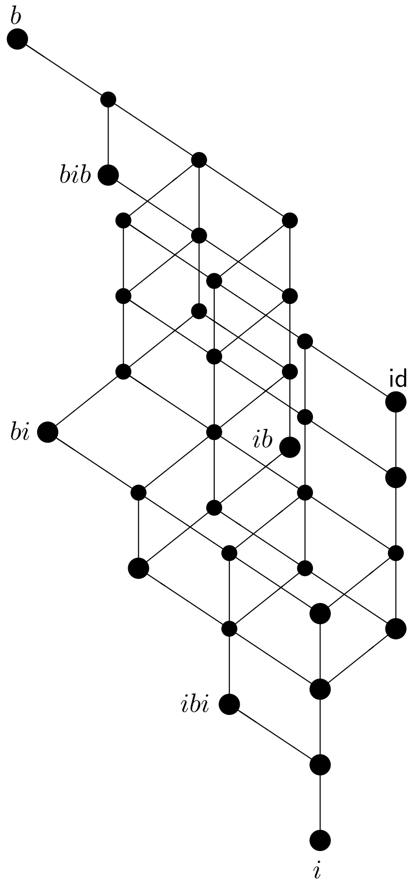


FIGURE 4.2. The 35 different operators produced from b, i, $\wedge$ and $\vee$.

It remains to find a topological space with a subset A on which all these operators act differently. Sherman gives the set

$$\left[\bigcup_{n\in\mathbb{N}}\left\{\frac{1}{n}\right\}\right] \cup \left[[2,4]\setminus\bigcup_{n\in\mathbb{N}}\left\{3+\frac{1}{n}\right\}\right]$$
$$\cup\left[(5,7]\cap\left(\mathbb{Q}\cup\bigcup_{n\in\mathbb{N}}\left(6+\frac{1}{2n\pi}, 6+\frac{1}{(2n-1)\pi}\right)\right)\right]$$

on which each of the 35 possible $\{b, i, \wedge, \vee\}$-operators are distinct under the usual topology on the reals. Here we instead take a systematic approach that also provides us with a finite example with a minimal number of points. The existence of both these examples is guaranteed by Theorem 4.1.

The approach is to construct a topological space around the 13 element ordered set in Figure 4.1 (a). Let X be the set of all elements in this ordered set along with one new element e. The natural order on $X\backslash\{e\}$ can be extended to e by setting $x \leq e$ for all x. For any element o of X denote by X_o the set $\{o' \in X : o' \leq o\}$ and Y_o to be the set $\{o' \in X : o' \nleq o\}$. Now notice that the open operators form a chain in the order on X so that if o_1 and o_2 are open operators and $o := \min\{o_1, o_2\}$ then we have $X_{o_1} \cap X_{o_2} = X_o$. Likewise for closed operators o_1, o_2 with $o := \max\{o_1, o_2\}$ we have $Y_{o_1} \cap Y_{o_2} = Y_o$. When o_1 is an open operator and o_2 is closed then $X_{o_1} \cap Y_{o_2}$ is empty except when $o_1 = ib$ and $o_2 = bi$ and then $X_{o_1} \cap Y_{o_2} = \{ib, \mathsf{id} \wedge ib\}$. This means we may take $\{\varnothing, X, X_i, X_{ibi}, X_{ib}, Y_{bi}, Y_{bib}, Y_b, \{ib, \mathsf{id} \wedge ib\}\}$ as a basis for a topology on X. The role of the element e is to ensure that the set $Y_b = \{x \in X : x \nleq b\}$ is not the empty set: we will require that to have proper closure.

It can now be routinely verified that each of the 35 operators in Figure 4.2 produces a distinct set when applied to the set X_{id}. Indeed, if $o_1, o_2, \ldots, o_n$ are operators from Figure 4.1 (a), and $o := o_1 \vee o_2 \vee \cdots \vee o_n$, then $o(X_{\mathsf{id}}) = X_{o_1} \cup \cdots \cup X_{o_n}$. For an example, consider the following:

$$\begin{aligned}((bi \wedge \mathsf{id}) \vee ib)(X_{\mathsf{id}}) &= (bi \wedge \mathsf{id})(X_{\mathsf{id}}) \cup ib(X_{\mathsf{id}}) \\ &= (bi(X_{\mathsf{id}}) \cap X_{\mathsf{id}}) \cup ib(X_{\mathsf{id}}) \\ &= (X_{bi} \cap X_{\mathsf{id}}) \cup i(X_b) \\ &= (\{bi \wedge \mathsf{id}, bi \wedge ib \wedge \mathsf{id}, ibi \wedge \mathsf{id}, i\}) \cup X_{ib} \\ &= X_{bi \wedge \mathsf{id}} \cup X_{ib}\end{aligned}$$

Hence all 35 possible operators act differently on the set X_{id}.

We now prove that amongst all spaces containing a subset on which each of the 35 operators in Figure 4.2 act differently, the space just given has a minimal number of points.

Proposition 4.3. *If $(X, \mathcal{T})$ contains a set A from which 35 distinct sets can be obtained by applying closure, interior, union and intersection, then $|X| \geq 14$. The bound $|X| = 14$ is attained in the above example.*

Proof. Let $(X, \mathcal{T})$ be a topological space with a subset A on which each of the 35 operators acts differently.

An element x of a lattice L is *join irreducible* if every finite set of elements ordered strictly below x in L have a join that is strictly below x. From Figure 4.2 it can be seen that in the lattice of $\{i, b, \wedge, \vee\}$-operators, the join irreducibles are precisely the $\{i, b, \wedge\}$-operators (as listed in the Hasse diagram in Figure 4.1 (a)). It is an elementary consequence of the distributivity of $\wedge$ over $\vee$ that whenever a join of some elements $o_1, \ldots, o_n$ is ordered above a join irreducible element o, then at least one of the elements o_i is itself ordered above o. Indeed, we have $o = o \wedge (o_1 \vee \cdots \vee o_n) = (o \wedge o_1) \vee \cdots \vee (o \wedge o_n)$ and join irreducibility then implies that at least one of the $o \wedge o_i$ is not strictly below o, whence at least one o_i is greater or equal to o. Thus for each $\{i, b, \wedge\}$-operator $o > i$, there is a point in $o(A)$ that does not appear in $o'(A)$ for any $\{i, b, \wedge\}$-operator o' not ordered above

o in Figure 4.1 (a). Also, the set $i(A)$ is non-empty (as it has a closure that strictly contains it). This shows that the largest $\{i, b, \wedge\}$-set, $b(A)$, contains at least 13 points—at least 1 in $i(A)$ and at least one extra element for each of the 12 $\{i, b, \wedge\}$-sets properly containing $i(A)$. However $b(A)$ cannot equal X as it has strictly smaller interior. Hence a minimum of 14 points is required. □

The Kuratowski $\{b, \wedge\}$ problem.
Clearly if we start from a single set, only 2 sets can be obtained by applying closure and intersection. In [**25**] however, a topological space is constructed with two subsets A and B, for which infinitely many subsets can be obtained by taking closures and intersections of the sets A and B; furthermore, the possible containments between the various sets generated are described. While the example in [**25**] is not complicated, it is not as simple as the space given in Example 4.2, which Sherman [**38**] has observed also contains two subsets—the sets of even numbers and of odd numbers—from which infinitely many others may be derived by taking intersections and closures (this example does contain identifications between the various operators that may not in general be present). By Theorem 4.1 there is also a pair of subsets of the real numbers with the usual topology that achieves this. We note that since closures distribute over finite unions, the corresponding Kuratowski $\{b, \vee\}$ problem is almost trivial: starting from a collection of n sets, one may produce a maximum of $3^n - 1$ new sets by taking closures and intersections. A family of subsets of $\mathbb{R}$ for which this bound is achieved is $\{(0, 1), (2, 3), \ldots, (2n-2, 2n-1)\}$.

The Kuratowski $\{ia, \wedge\}$ and $\{ia, \vee\}$ problems.
The derived operator ia is usually called the exterior of a set but can also be thought of as a kind of *pseudo-complement* on a topological space: it assigns to each subset A of a topological space $(X, \mathcal{T})$, the largest member B of $\mathcal{T}$ for which $A \cap B = \varnothing$. A system of algebraic structures related to this operator has been studied in [**26**] and there it is shown that the number of sets obtainable from a single subset of a topological space using ia and $\wedge$ is equal to 10. The standard example of a Kuratowski 14-set given in [**44**, Example 32, part 9] (and in the introduction of this article) gives an example of a subset of the reals with this property. The ordering of 10 operators is given in Figure 4.3 (a). Here the operator 0 is a constant operator assigning every set the value $\varnothing$ and likewise the operator 1 assigns every set the value X.

It is easy to extend this problem to include the binary operator $\vee$. Let o_1, o_2,..., o_n be incomparable $\{ia, \wedge\}$-operators and consider the application of ia to the join $o_1 \vee o_2 \vee \cdots \vee o_n$. We get $ia(o_1 \vee o_2 \vee \cdots \vee o_n) = iao_1 \wedge iao_2 \wedge \cdots \wedge iao_n$. Since the ten $\{ia, \wedge\}$ operators are closed under applications of ia and of $\wedge$, we have $iao_1 \wedge iao_2 \wedge \cdots \wedge iao_n = o$ for some $\{ia, \wedge\}$-operator o. Hence every set obtainable from a given set A by taking ia, intersections and unions is equal to a union of some of the at most 10 sets obtained from A using ia, and intersections. While we again omit the details, one routinely arrives at further 26 possible operators in addition to the ten $\{ia, \wedge\}$-operators. To find an example on which all 36 of these potentially distinct operators act differently, take the nine $\{ia, \wedge\}$-operators other than 0 as the points and take as a basis for a topology, the downsets of the five open $\{ia, \wedge\}$-operators (under the order given in Figure 4.3). As in Proposition 4.3,

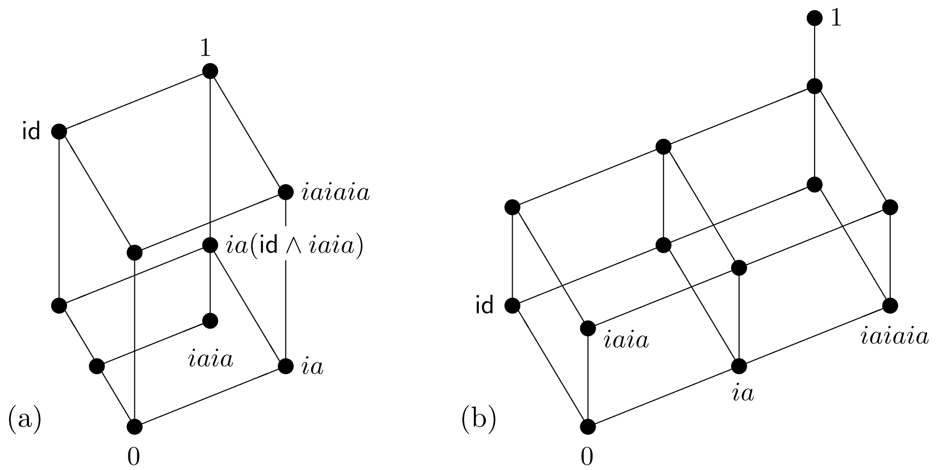


FIGURE 4.3. The 10 operators produced from ia and $\wedge$ and the 13 produced from ia and $\vee$.

this space is routinely seen to have a minimal number of points. Summarising we have the following.

Proposition 4.4. *The number of possible sets obtainable from a single set by taking intersections, unions and interiors of complements is* 36. *Any space that contains a set from which* 36 *possibilities can be obtained in this way has at least* 9 *points. There is a* 9 *point space on which this occurs.*

We can also extract a solution to the Kuratowski $\{ia, \vee\}$-problem from this argument. There are 13 operators in this case and the corresponding order on these is given in Figure 4.3 (b). The example of [**44**, Example 32, part 9] again suffices to show that each of these operators are distinct on the space of reals with the usual topology.

Kuratowski boundary problems.
Let $f(A) := b(A) \cap ba(A)$ denote the boundary of the subset A of a topological space. Despite its natural topological connections, results concerning the behaviour of f in conjunction with subsets of $\{a, b, i\}$ are neither well known nor easily accessible. There appear to be three places where the problem is studied in some detail. The first is in Zarycki [**50**], where the Kuratowski $\{a, f\}$-problem is shown to have a solution of 6. This is extended substantially by the work of Soltan [**43**]. In fact a slightly more general notion than topological closure is considered in [**43**] (see Subsection 4.2 for other variants of this type), but with only minor modifications one obtains the following table of results. The columns of this table indicate the operators being considered, while the number is the maximal value of the corresponding generalised k-number (equivalent cases are omitted, while $\{a, b\}$ and $\{b, i\}$ are included for completeness).

{b,f}	{a,f}	{b,i}	{i,f}	{a,b}	{b,i,f}	{a,b,f}
5	6	7	8	14	17	34

The last of these columns gives the most general result and so a description of the distinct operators in this case will give all other columns as well. We will not give a full proof of Soltan's result, but note that the upper bound of 34 arises from the existing properties of a, b and the following identifications that can be proved directly from the definition with only a little work (recall that 0 and 1 are operators on the space $(X, \mathcal{T})$ assigning each set the value $\varnothing$ and X respectively):

$$f^2 = f^3, \quad fa = bf = f, \quad fb = f^2b,$$
$$ifb = 0, \quad fbi = fibi, \quad f0 = b0 = 0 = f1,$$
$$a0 = b1 = 1.$$

For an example, consider the action of ifb on a subset A of a topological space. Using the definition of f and the usual properties of closures and interiors, one finds that $ifb(A) = ib(A) \cap abib(A)$ and then, because $bib(A) \supseteq ib(A)$, we have $ifb(A) = \varnothing = 0(A)$.

Using the above identifications and associativity, one can easily prove that any $\{a, b, f\}$-operator is equivalent to one of the standard Kuratowski operators, or one of the following operators:

$$f, f^2, fb, fba, fbab, fbaba, af, af^2, afb, afba, afbab,$$
$$afbaba, baf, abaf, babaf, ababaf, fbaf, afbaf, 0, 1$$

(note that it is the Kuratowski operator id, not 1, that is the identity element of the corresponding operator monoid). Each of these is written in the shortest possible form in terms of the operators $\{a, b, f\}$. These operators can be proved distinct by starting with the Kuratowski set given at the start of this paper, or even from a simplified example such as $(0, 1) \cup (1, 2) \cup (\mathbb{Q} \cap (2, 3)) \cup \{4\}$. We note that as well as these results, Soltan also finds the minimal size that a generalised space can have in order that it contain a subset on which the various maxima in the table can be achieved (recall that the closure in [**43**] is not necessarily topological).

The third place where boundaries are used in conjunction with closures, interiors and complements is in [**34**], where the first 5 values of the above table are again obtained. For each of these cases, a description is given of what properties a subset of a space must have in order to achieve this bound (extending the results of [**30**]).

Zarycki [**50**] also examined derived operator $z := \mathsf{id} \wedge ba$ (equivalently, $z = \mathsf{id} \wedge f$). Despite apparent similarities to the the frontier operator f, Zarycki showed that the solution to the Kuratowski $\{z, a\}$-problem is infinite. As an example, notice the operators $z, zaz, zazaz, \ldots$ all act differently on the set $\{1, 3, 5, \ldots\}$ of Example 4.2.

4.2. Non-topological variants. The Kuratowski complement-closure problem holds in a considerably more general setting than topological spaces. For example, the proof of this result does not require that closure distribute over finite unions of sets. Variations of this relaxed version of the Kuratowski theorem have been investigated by a large number of authors. These investigations fall into two groups: the general properties of such abstractions; the behaviour of various concrete examples of non-topological closures.

Perhaps the first investigation into Kuratowski-like phenomena amongst the abstract theory of non-topological closures is the work of Hammer [**21**]. Hammer investigates necessary identities between generalised closure and interior operators. Here for example, a closure operator on the subsets of a set X is defined to be an isotonic, idempotent operator f satisfying $f \geq \mathsf{id}$. Some problems involving generalised closures that are posed in [**21**] are solved in [**13**] and [**40**]. Further investigations into generalised closures can be found in the work of Soltan [**42, 43**]. In [**42**], relationships between the number of sets obtainable using (generalised) closure and complement and the number obtainable using generalised closure and its corresponding interior are investigated. See the discussion of Kuratowski boundary problems in Subsection 4.1 for some of the results of [**43**].

The Kuratowski-type behaviour of generalised notions of closure are rediscovered again in Peleg [**36**]. Peleg also considers several interesting concrete examples such as the complement operator and Kleene closure on formal languages: the maximum number of languages obtainable is 14 and it is commented that this bound is achieved by the language $\{a, aab, bbb\}$.

A natural generalised closure operator called the *convex hull* is examined by Koenen [**28**]. This operator associates with each subset S of $\mathbb{R}^n$ the intersection of all convex sets containing S. It is shown that at most 10 sets can be produced from a subset of $\mathbb{R}^n$ by taking convex hulls and complements, and that this bound is achieved by the subset $\{(0, 1)\} \cup \{(x, y) : y \leq 0 \text{ and } x^2 + y^2 \neq 0\}$ in $\mathbb{R}^2$ (the bound of 10 for $\mathbb{R}^2$ is stated without proof in [**21**]). The corresponding Kuratowski monoid is extremally disconnected in our notation.

One of the most important of the generalised closures considered in the literature are those arising from the study of binary relations. The first paper concerning versions of the Kuratowski closure-complement theorem for operators on binary relations appears to be that of Graham, Knuth and Motzin [**19**]. It is shown that the number of distinct relations obtainable from a given binary relation by taking complements and transitive closures is at most 10 (here it is convenient to view a binary relation on a set X as a subset of the cartesian product $X \times X$). They also show that the largest possible number of relations obtained from a given relation by taking complements, transitive closures and reflexive closures is 42. The transitive closure and complement result is further developed in [**17**]. A larger class still of operators on relations is considered by Fishburn in [**16**]. Here it is shown that the maximum number of relations obtainable from a single binary relation by applying the operators of asymmetrisation, complementation, dualisation, symmetrisation, and transitive closure is 110; again it is shown that there is a relation that achieves this bound.

A natural non-topological closure from algebra occurs when one considers the subalgebra generated by a subset of an algebra. Closures of this type are briefly examined in [**1**] (see also in Subsection 2.1 of this article). In [**1**] it is shown for example that there is an abstract algebra with 6 elements and with a subset A from which 14 distinct sets can be obtained by taking complements and generating subalgebras (that no more than 14 sets can be obtained in this way follows from the proof of the Kuratowski theorem).

As a final example of interest (also of algebraic nature) we note a result of Pigozzi [**37**]. Here the standard class operators of universal algebra are considered; that is, closure operators on classes of algebras of a particular type. There are three operators: $\mathbb{S}$, which corresponds to taking all isomorphic copies of subalgebras of a given class; $\mathbb{H}$, which corresponds to taking all homomorphisms of algebras in a given class; and $\mathbb{P}$, which corresponds to taking all direct products of algebras in a given class. For example it is well known that a class of algebras is equationally definable if and only if it is closed under the operator $\mathbb{HSP}$. Pigozzi shows that at most 18 different operators can be obtained by combinations of these, the largest being the operator $\mathbb{HSP}$. Moreover, the bound of 18 is achieved over families of groups. A survey of results concerning operators obtained from $\{\mathbb{H}, \mathbb{S}, \mathbb{P}\}$ is given in [**14**]. In [**35**] analogous questions are investigated for operator monoids generated by other sets of class operators—such as $\{\mathbb{H}, \mathbb{S}, \mathbb{P}, \mathbb{P}_S\}$ (where $\mathbb{P}_S$ indicates subdirect product). The particular case of $\{\mathbb{H}, \mathbb{S}, \mathbb{P}, \mathbb{P}_S\}$ is studied further in [**47**] where it is shown that at most 22 operators can be formed and that this bound is again achieved over families of groups.

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B.J. Gardner
Discipline of Mathematics
University of Tasmania
Private Bag 37
Hobart, Tas 7001
AUSTRALIA
gardner@hilbert.maths.utas.edu.au

Marcel Jackson
Department of Mathematics and Statistics
La Trobe University
Vic 3086
AUSTRALIA
m.g.jackson@latrobe.edu.au