# ON TURBULENCE MODELLING AND THE TRANSITION FROM LAMINAR TO TURBULENT FLOW

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## Abstract

Fluid turbulence is often modelled using equations derived from the Navier–Stokes equations, perhaps with some semi-heuristic closure model for the turbulent viscosity. This paper considers a possible alternative hypothesis. It is argued that regarding turbulence as a manifestation of non-Newtonian behaviour may be a viewpoint of at least comparable validity. For a general description of nonlinear viscosity in a Stokes fluid, it is shown that the flow patterns are indistinguishable from those predicted by the Navier–Stokes equation in one- or two-dimensional geometry, but that fully three-dimensional flows differ markedly. The stability of linearized plane Poiseuille flow to three-dimensional disturbances is then considered, in a Tollmien–Schlichting formulation. It is demonstrated that the flow may become unstable at significantly lower Reynolds numbers than those expected from Navier–Stokes theory. Although similar results are known in sections of the rheological literature, the present work attempts to advance the philosophical viewpoint that turbulence might always be regarded as a non-Newtonian effect, to a degree that is dependent only on the particular fluid in question. Such an approach could give a more satisfactory account of the underlying physics.

## 1. Introduction

It seems fair to state that the understanding and modelling of turbulence still remains the stand-out unsolved question of fluid mechanics. The problem has received an enormous amount of attention over the past century, and from some of the most eminent scholars in the subject, and the literature on the topic is enormous; consequently, any claim for a new approach to the topic is inevitably risky. Nevertheless, it remains the case that equations used to describe turbulence often meet with only mixed success, in that, while being effective for a particular application to

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which they have been tuned, they are considerably less able to model a different flow situation, for which they have not been explicitly calibrated. In addition, the physical processes underpinning turbulent behaviour are still not well resolved.

Turbulent flows are inherently difficult and complicated, because they are unavoidably three-dimensional; additionally, they are fundamentally unsteady, and are characterized by velocities that fluctuate in an apparently random manner. They also involve enhanced rotational motion, with eddies appearing at a variety of length scales, and a cascade of energy down to very small characteristic (Kolmogorov) lengths, at which energy is dissipated efficiently by viscous effects. These processes are discussed in detail in the monograph by Davidson [6]. In the modelling of turbulence, it is generally assumed that the behaviour is described by the Navier–Stokes equations of viscous flow, and often with the continuity equation appropriate for incompressible fluids. This approach is articulated in the text by Batchelor [2], for example, and is also stated as a certainty on the first page of Chandrasekhar's Lectures on the topic, edited by Spiegel [32]. The review paper by Eckhardt [8] on open problems in turbulent pipe flow likewise starts from this premise.

Assuming, then, that the Navier–Stokes equations are indeed the correct model for fluid flow including turbulence, it is natural to use them to study the transition to unstable or turbulent flow, from a simple base flow such as shearing flow between parallel plates. Early work on this problem was carried out by Squire [33], partly in response to the famous 1895 paper of Osborne Reynolds [27], in which he claimed that turbulent eddies for flow in a (circular) pipe would occur for a Reynolds number of about 2000. Squire's [33] analysis of the Navier–Stokes equations led him to conclude that fully three-dimensional small disturbances to a simple base shearing flow are more stable than their two-dimensional counterparts, a result known as *Squire's theorem* in some of the rheological literature (see Larson [17] and Jerome and Chomaz [15]); this terminology is also used by Drazin and Reid [7, p. 155]. Squire [33] additionally determined that several simple shear flows remained stable for all Reynolds numbers. Orszag [22] studied the stability of plane Poiseuille flow for the Navier–Stokes equation, and carried out an accurate solution of the Orr–Sommerfeld equation [7, p. 156] that describes the behaviour of small-amplitude disturbances. His methods were based on the use of series of Chebyshev polynomials and the QR algorithm for the determination of eigenvalues. He found that plane Poiseuille flow became (weakly) unstable at a Reynolds number Re = 5772.22, over a very narrow interval of wavenumbers. Further refinements of this result have since been undertaken, and some of these are discussed by Drazin and Reid [7].

These stability calculations for simple solutions of the Navier–Stokes equations, such as plane Poiseuille flow, clearly indicate a large difference between the Reynolds number at which instability is predicted to occur, and those actually measured in experiments. As a result, some degree of dissatisfaction with linear stability theory has arisen. An alternative "stability without eigenvalues" [35] approach has been sought, in the context of the Navier–Stokes equations. Waleffe [36] has argued that the transition to turbulence is characterized more by the appearance of large-scale

coherent structures near the wall, and experimental observation of such fluid behaviour has indeed been reported by Hof et al. [13]. Purely nonlinear stability analyses have also been undertaken [5].

From the more practical aspect of computing turbulent flow in engineering applications, the most usual approach taken is the one originally suggested by Reynolds [27]. The fluid velocity vector $\mathbf{q}$ is regarded as consisting of a slowly varying (average) component $\bar{\mathbf{q}}$ and a rapidly fluctuating contribution $\mathbf{q}'$ with expectation value zero, so that $\mathbf{q} = \bar{\mathbf{q}} + \mathbf{q}'$. This is substituted into the Navier–Stokes equations, which are then averaged in time, to produce the famous Reynolds-averaged Navier–Stokes equations. Due to the nonlinearity of the system, however, averaged quantities of the form $\overline{q_i' q_j'}$ also appear in the system, for which no obvious equation exists. This is the famous "closure problem" of turbulence, and is reviewed by Sreenivasan [34]. Heuristic arguments based on dimensional analysis are used to model these averaged quantities in terms of more computationally accessible slowly varying variables; in more modern "$k$-epsilon" models of turbulence, these variables are even assumed to obey their own transport-type equations. This approach is not discussed further here, but nonlinear models of this type are presented by Speziale [31].

In addition to these pragmatic engineering approaches to the closure problem of turbulence, there is now a vast literature on statistical theories. The aim is essentially the same; that is, advanced statistical methods that have their origin in statistical field theory are used to seek equations that are closed for some property of the flow. An early text is that of Batchelor [2]. Kraichnan [16] pioneered the use of renormalization group methods, and these ideas have been developed and extended by Martin et al. [19] and Jensen [14], for example. Again, the starting point for these investigations is the incompressible Navier–Stokes equations, which are taken as a satisfactory description of the fluid behaviour. A comprehensive review of these approaches is given in the article by McComb [21], and a more recent appraisal of the extensive literature on these techniques is presented in the text by Davidson [6]. These statistical methods are not discussed further here, although they have recently been applied to problems in geophysical flows, in which the underlying behaviour is assumed to be described by a barotropic vorticity equation closely related to the Navier–Stokes equation. A review of this work is presented by Frederiksen and O'Kane [10], and Frederiksen [9] has used this approach to obtain statistical dynamical closures for quasi-geostrophic flows of interest in oceanography.

In his 1883 paper on what is now known as the Reynolds number, Osborne Reynolds [26] stated that: "The results of this investigation have both a practical and a philosophical aspect. … The results as viewed in their philosophical aspect were the primary object of the investigation". The present paper is written with precisely the same aims in mind. The focus here is primarily philosophical, with the intention to examine the alternative hypothesis that turbulence might be regarded as a physical phenomenon *not* governed by the Navier–Stokes equations alone, but rather that it reflects a non-Newtonian constitutive law. It is accepted here that such a hypothesis will be controversial, and certainly not universally accepted; and the aim, therefore,

is to test whether such a proposition might be capable of producing results of at least equal value to those currently available through some of the techniques discussed in this section.

A general nonlinear relation between stress and strain rate is investigated in Section 2. This leads to a version of the well-known Reiner–Rivlin equations, proposed by Reiner [24] and referred to by him as providing a theory of "dilatancy", and then developed by Rivlin [28, 29]. In that section a key theorem is proved, showing that the governing equation reduces to one that gives the identical flow patterns to the Navier–Stokes equations, for two-dimensional flow. Section 4 studies the relationship between this Reiner–Rivlin type equation and the Navier–Stokes equation in regard to the generation of vorticity. In fully three-dimensional flow, the Reiner–Rivlin model has additional nonlinear terms that generate vorticity, and so, if this equation does indeed provide a model for general turbulence, these extra terms would then largely be responsible for effects such as vortex stretching in three dimensions. A comparison is then made of the linearized stability problem in Section 5, for two-dimensional Poiseuille flow in both the Reiner–Rivlin model and its Navier–Stokes counterpart. It is found that unstable behaviour may occur for lower Reynolds numbers than predicted by purely Navier–Stokes theory, dependent on the value of the coefficient of the second, nonlinear, viscosity term. This is in accordance with results obtained for highly rheological fluids in the investigations of Wilson et al. [37]. In addition, higher wavenumber disturbances are more unstable. Finally, an expanded discussion of this alternative hypothesis concerning turbulence modelling is given in Section 6.

## 2. Stokes fluids

The approach outlined in this section is well known in rheology, and is presented in texts such as Aris [1]. Accordingly, only a relatively brief overview is needed here. It is important to state that exotic rheological behaviour is not of interest here; rather, the intention is to consider the possibility that, in fully developed turbulent flow of a quite general fluid, it may *not* be appropriate to assume the linear (Newtonian) relation between the stress tensor $\mathbf{T}$ and the strain-rate tensor $\mathbf{D}$ that underlies Navier–Stokes theory. Instead, a general nonlinear relationship is needed, to account for the large strain rates experienced by the fluid in such circumstances. The resulting equation is then a type of Reiner–Rivlin equation, as discussed by Aris [1].

The fluid velocity vector is represented in Cartesian form by the vector $\mathbf{q}$ with components written as $(q_1, q_2, q_3) = (u, v, w)$, and the coordinate variables are $(x_1, x_2, x_3) = (x, y, z)$. For a fluid of density $\rho$ and subject to body force vector $\mathbf{f}$ per mass, Cauchy's momentum equation is

$$\rho \left[ \frac{\partial q_i}{\partial t} + \mathbf{q} \cdot \nabla q_i \right] = \rho f_i + \frac{\partial T_{ji}}{\partial x_j}, \tag{2.1}$$

and may be found in Batchelor [3, p. 137] or Mase [20, p. 128]. Here, summation over repeated indices is assumed, according to the Einstein convention. From Aris [1, Section 5.13], the conservation of angular momentum for a nonpolar fluid then yields the requirement that the stress tensor $\mathbf{T}$ is symmetric.

It is customary to write $\mathbf{T} = -p\mathbf{I} + \mathbf{S}$, in which the pressure is $p$, $\mathbf{I}$ is the identity matrix and $\mathbf{S}$ is a deviatoric stress tensor (see Mase [20, Section 2.14]). For a Stokes fluid, it may be assumed that $\mathbf{S}$ is an analytic function of the strain-rate tensor $\mathbf{D}$, which is symmetric and has components

$$D_{ij} = \frac{1}{2}\left[\frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i}\right]. \tag{2.2}$$

In general, therefore, $\mathbf{S}$ can be represented as an infinite series in powers of $\mathbf{D}$, but a consequence of the Cayley–Hamilton theorem (see Golubitsky and Dellnitz [11, p. 464]) is that powers higher than the second need not be considered, since $\mathbf{D}$ is a $3 \times 3$ matrix. Thus a general description of material nonlinearity for a nonpolar Stokes fluid is encapsulated in a constitutive relation of the type

$$S_{ij} = K_{ijpq}D_{pq} + L_{ijpq}D_{p\ell}D_{\ell q}. \tag{2.3}$$

Again, summation over repeated indices is assumed, and the fourth-order tensors $\mathbf{K}$ and $\mathbf{L}$ are constants. Following Aris [1], symmetry and isotropy of the deviatoric stress tensor in equation (2.3), given the symmetry of the quantity $\mathbf{D}$ in (2.2), requires the fourth-order coefficient tensors to take the forms

$$K_{ijpq} = \lambda\delta_{ij}\delta_{pq} + \mu(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}),$$
$$L_{ijpq} = \sigma\delta_{ij}\delta_{pq} + \tau(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}),$$

in which $\delta_{ij}$ is the Kronecker delta symbol, having the value 1 if $i = j$ and 0 otherwise. As a result, the deviatoric stress tensor (2.3) can be written as

$$S_{ij} = \lambda\delta_{ij}D_{kk} + 2\mu D_{ij} + \sigma\delta_{ij}(D_{k\ell}D_{k\ell}) + 2\tau D_{i\ell}D_{\ell j}$$

and is symmetric, in view of the symmetry in equation (2.2).

In Cartesian form, a general equation for fluid flow, that expresses material nonlinearity in a nonpolar isotropic substance, is therefore derived from the Cauchy equation (2.1) in the form

$$\rho\left[\frac{\partial q_i}{\partial t} + \mathbf{q} \cdot \nabla q_i\right] = \rho f_i - \frac{\partial p}{\partial x_i} + \lambda\frac{\partial}{\partial x_i}(D_{kk})$$
$$+ 2\mu\frac{\partial D_{ji}}{\partial x_j} + \sigma\frac{\partial}{\partial x_i}(D_{k\ell}D_{k\ell}) + 2\tau\frac{\partial}{\partial x_j}(D_{j\ell}D_{\ell i}). \tag{2.4}$$

The four constants $\lambda$, $\mu$, $\sigma$ and $\tau$ are viscosity coefficients. For an incompressible fluid, the continuity equation div $\mathbf{q} = 0$ gives $D_{kk} = 0$, so that in that case, the second Lamé constant $\lambda$ can be set to zero with no loss of generality. The quantity $\mu$ is the usual dynamic viscosity, so that $\sigma$ and $\tau$ measure the extent of material nonlinearity. This governing equation (2.4) is essentially the well-known Reiner–Rivlin equation.

### 3. Viscometric and planar flows

**3.1. Viscometric flows**   In unidirectional flow, the velocity vector $\mathbf{q}$ takes the form $(q_1, 0, 0) = (u, 0, 0)$. Only the incompressible flow case is considered here, for which the continuity equation becomes simply $\partial u/\partial x = 0$. As a result, the full system (2.4) simplifies very substantially to

$$
\begin{aligned}
\rho \frac{\partial u}{\partial t} &= \rho f_X - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \\
0 &= \rho f_Y - \frac{\partial p}{\partial y} + \frac{1}{2}\sigma\frac{\partial}{\partial y}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right] \\
&\quad + \frac{1}{2}\tau\left[\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\frac{\partial u}{\partial y}\right)\right] \\
0 &= \rho f_Z - \frac{\partial p}{\partial z} + \frac{1}{2}\sigma\frac{\partial}{\partial z}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right] \\
&\quad + \frac{1}{2}\tau\left[\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\frac{\partial u}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right)^2\right].
\end{aligned}
\tag{3.1}
$$

The first equation in this system (3.1) is now *linear*, and indeed it is identical to the same equation obtained from the Navier–Stokes equation in the same geometry. As a consequence, all the usual unidirectional viscometric flows may be obtained, as in Batchelor [3, Section 4.2]. However, while the flows are identical, the pressures are not.

As an illustration, consider the classical two-dimensional Poiseuille flow, in which impermeable flat plates are present on the two planes $z = -H$ and $z = H$. There are no body forces ($f_X = f_Y = f_Z = 0$) and the fluid is driven through the gap between the plates by an imposed constant pressure gradient $\partial p/\partial x = -G$. Then the first equation in the system (3.1) gives the well-known velocity profile

$$
u(z) = \frac{G}{2\mu}(H^2 - z^2)
\tag{3.2}
$$

(see Batchelor [3, p. 182]). However, the remaining two conditions in (3.1) give the corresponding pressure

$$
p(x, z) = -Gx + \frac{1}{2}(\sigma + \tau)\left(\frac{G}{\mu}\right)^2 z^2.
\tag{3.3}
$$

Flows of this type (3.2) are used experimentally in viscometry, where the fluid viscosity is calculated based on the measured speeds. For non-Newtonian fluids, however, calculating the viscosity parameters may be difficult, as discussed by Shaqfeh [30]. Clearly, from equation (3.3), the pressure in such a fluid rises near the walls $z = \pm H$, and this may permit measurement of some of the nonlinear viscosity coefficients.

**3.2. Planar flows**   It is instructive now to consider the form of the Reiner–Rivlin type equation (2.4) in two-dimensional flow geometry, where the velocity components

are $(q_1, q_2, q_3) = (u, v, 0)$ and there is no dependence on the third coordinate $x_3$. For incompressible fluids, the continuity equation is simply

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

and as a result, the planar momentum equation (2.4) can be written in the vector form

$$\rho \left[ \frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} \cdot \nabla \mathbf{q} \right] = \rho \mathbf{f} - \nabla p + \mu \nabla^2 \mathbf{q} + 2(\sigma + \tau) \nabla \chi, \tag{3.4}$$

after some considerable algebra. Here, the symbol $\nabla$ is simply the two-dimensional operator $(\partial/\partial x, \partial/\partial y)$ and similarly for the two-dimensional operator $\nabla^2$. In equation (3.4) the intermediate scalar quantity

$$\chi = \frac{1}{4} \left[ 4 \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \right] \tag{3.5}$$

has been defined for convenience.

This at once leads to the following result:

THEOREM 3.1. *In unidirectional and planar flow, the flow patterns predicted by the Reiner–Rivlin type equation* (2.4) *are indistinguishable from those obtained from the Navier–Stokes equation. The additional nonlinear viscosity terms only influence the pressure in the fluid.*

PROOF. The key observation is that, in the two-dimensional case (3.4), the additional viscosity terms only appear as the gradient of the scalar function $\chi$ defined in equation (3.5). Consequently, by defining the effective pressure

$$\Pi_{\text{eff}} = p - 2(\sigma + \tau)\chi, \tag{3.6}$$

the two-dimensional velocity vector $\mathbf{q}$ now satisfies the two-dimensional Navier–Stokes equation, but with pressure given by (3.6). □

Since the nonlinear viscosity terms only enter equation (3.4) as the gradient of $\chi$, taking the vector curl immediately gives precisely the same vorticity equation as for two-dimensional Navier–Stokes flow. This provides an alternative verification of the first claim in Theorem 3.1.

## 4. Flow comparison with Navier–Stokes theory

A consideration of the vorticity vector $\zeta = \text{curl } \mathbf{q}$ provides a more focused comparison of the flow predicted by the Reiner–Rivlin type equation (2.4) with the corresponding predictions of Navier–Stokes theory. In Cartesian coordinates, the vorticity may be written

$$\zeta_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} q_k \tag{4.1}$$

in which the Levi-Cività symbol $\epsilon_{ijk}$ is zero if any of the three subscripts are equal; it also takes the values 1 if the subscripts are in cyclic order and $-1$ if these are

ordered anti-cyclically (see Mase [20, Section 1.16]). Now the curl is taken of the entire equation of motion (2.4). Recalling that Aris [1, Section 6.2] described the quantity of algebra required to use a Reiner–Rivlin equation as "appalling", the task of taking its vector curl is Herculean, and so only a summary is given here. It is relatively straightforward to derive the vorticity equation in the form

$$
\rho\left[\frac{\partial \zeta_i}{\partial t} + (\mathbf{q} \cdot \nabla)\zeta_i - (\zeta \cdot \nabla)q_i\right]
$$

$$
= \mu\nabla^2\zeta_i + \frac{1}{2}\tau\epsilon_{ijk}\frac{\partial}{\partial x_j}\left[\frac{\partial q_m}{\partial x_\ell}\frac{\partial}{\partial x_m}\left(\frac{\partial q_\ell}{\partial x_k}\right) + \frac{\partial q_m}{\partial x_\ell}\frac{\partial}{\partial x_m}\left(\frac{\partial q_k}{\partial x_\ell}\right)\right.
$$

$$
\left. + (\nabla^2 q_\ell)\frac{\partial q_\ell}{\partial x_k} + \frac{\partial q_\ell}{\partial x_m}\frac{\partial}{\partial x_m}\left(\frac{\partial q_\ell}{\partial x_k}\right) + (\nabla^2 q_\ell)\frac{\partial q_k}{\partial x_\ell} + \frac{\partial q_\ell}{\partial x_m}\frac{\partial}{\partial x_m}\left(\frac{\partial q_k}{\partial x_\ell}\right)\right]. \quad (4.2)
$$

It follows after a great deal of algebra that the first and the fourth terms in the coefficient of $\tau$ in equation (4.2) are each separately zero. There are now no further apparent simplifications to this difficult vorticity equation, and after a great deal more algebra it may be expressed in the final form

$$
\rho\left[\frac{\partial \zeta}{\partial t} + (\mathbf{q} \cdot \nabla)\zeta - (\zeta \cdot \nabla)\mathbf{q}\right]
$$

$$
= \mu\nabla^2\zeta + \frac{1}{2}\tau\left[\sum_{j=1}^{3}\nabla(\nabla^2 q_j) \times \nabla q_j\right] + \frac{1}{2}\tau\nabla \times \left[\sum_{j=1}^{3}\frac{\partial \mathbf{q}}{\partial x_j}(\nabla^2 q_j)\right]
$$

$$
+ \tau\nabla \times \left[\sum_{j=1}^{3}(\nabla q_j \cdot \nabla)\frac{\partial \mathbf{q}}{\partial x_j}\right]. \quad (4.3)
$$

Although complicated, this vorticity equation preserves the traditional advantage that the terms involving pressure disappear, as a result of a vector identity, so that the vorticity equation (4.3) focuses purely on aspects of the flow. In addition, the terms involving the second nonlinear viscosity coefficient $\sigma$ in equation (2.4) also vanish, since they are pure gradient terms and are eliminated by the vector curl operation. Thus the parameter $\sigma$ in (2.4) only affects pressure, and might therefore be set to zero without serious loss of generality.

When the flow is two-dimensional, so that $q_3 = 0$ and $\partial/\partial x_3 \equiv 0$, there is only the single nonzero component $\zeta_3$ in the vorticity function (4.1). In that case, it is again pertinent to observe that all the terms in (4.3) involving the second vorticity coefficient $\tau$ become precisely zero, so that the standard scalar vorticity equation

$$
\frac{\partial \zeta_3}{\partial t} + (\mathbf{q} \cdot \nabla)\zeta_3 = \frac{\mu}{\rho}\nabla^2\zeta_3
$$

for the single component $\zeta_3$ is recovered (see Batchelor [3, p. 268]). This is consistent with Theorem 3.1. Thus the additional nonlinear viscosity terms in (4.3) involving the parameter $\tau$ only have an effect when the flow is fully three-dimensional, and perhaps even account for the additional vorticity observed in turbulent flow behaviour.

## 5. Stability of plane Poiseuille flow

In this section, a classical Orr–Sommerfeld stability analysis is undertaken, to see the effect of the additional nonlinear viscosity terms in equation (2.4) on fully three-dimensional small disturbances to a simple basic flow. In order to make the volume of algebra manageable, the background flow is taken to be a steady-state unidirectional solution of the viscometric type.

The velocity components and the pressure are expanded as

$$
\begin{aligned}
u(x, y, z, t) &= U_0(z) + \epsilon u_1(x, y, z, t) + O(\epsilon^2), \\
v(x, y, z, t) &= \epsilon v_1(x, y, z, t) + O(\epsilon^2), \\
w(x, y, z, t) &= \epsilon w_1(x, y, z, t) + O(\epsilon^2), \\
p(x, y, z, t) &= P_0(z) + \epsilon p_1(x, y, z, t) + O(\epsilon^2).
\end{aligned}
\tag{5.1}
$$

Here, $U_0(z)$ is the flow speed of the background unidirectional flow in the $x$-direction, and $P_0(z)$ is the corresponding pressure for this flow. The dimensionless parameter $\epsilon$ is a small constant.

These perturbation expansions (5.1) are substituted into the Reiner–Rivlin type equation (2.4) and the incompressible continuity equation div $\mathbf{q} = 0$, and only terms of first order in the parameter $\epsilon$ are retained. The continuity equation gives

$$
\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0.
\tag{5.2}
$$

The three components of the linearized momentum equation (2.4) are lengthy, as is to be expected in view of the complicated nature of this system. They may be simplified to some extent using the linearized continuity condition (5.2), and the $x$-component gives

$$
\rho \left[ \frac{\partial u_1}{\partial t} + U_0(z) \frac{\partial u_1}{\partial x} + U_0'(z) w_1 \right] = -\frac{\partial p_1}{\partial x} + \mu \nabla^2 u_1 + (\sigma + \tau) U_0'(z) \left[ \frac{\partial^2 w_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial x \partial z} \right]
$$
$$
- \tau U_0''(z) \frac{\partial v_1}{\partial y} + \frac{1}{2} \tau U_0'(z) \left[ \frac{\partial^2 w_1}{\partial y^2} - \frac{\partial^2 v_1}{\partial y \partial z} \right].
\tag{5.3}
$$

The $y$-component of the momentum equation linearizes to

$$
\rho \left[ \frac{\partial v_1}{\partial t} + U_0(z) \frac{\partial v_1}{\partial x} \right]
$$
$$
= -\frac{\partial p_1}{\partial y} + \mu \nabla^2 v_1 + \sigma U_0'(z) \left[ \frac{\partial^2 w_1}{\partial x \partial y} + \frac{\partial^2 u_1}{\partial y \partial z} \right] + \frac{1}{2} \tau U_0''(z) \left[ \frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x} \right]
$$
$$
+ \frac{1}{2} \tau U_0'(z) \left[ \frac{\partial^2 w_1}{\partial x \partial y} + 2 \frac{\partial^2 v_1}{\partial x \partial z} + \frac{\partial^2 u_1}{\partial y \partial z} \right],
\tag{5.4}
$$

and the final $z$-component of momentum gives the linearized equation

$$\rho\left[\frac{\partial w_1}{\partial t} + U_0(z)\frac{\partial w_1}{\partial x}\right]$$

$$= -\frac{\partial p_1}{\partial z} + \mu\nabla^2 w_1 + (\sigma + \tau)U_0'(z)\left[\frac{\partial^2 w_1}{\partial x\partial z} + \frac{\partial^2 u_1}{\partial z^2}\right] + (\sigma + \tau)U_0''(z)\left[\frac{\partial w_1}{\partial x} + \frac{\partial u_1}{\partial z}\right]$$

$$+ \frac{1}{2}\tau U_0'(z)\left[\frac{\partial^2 u_1}{\partial y^2} - \frac{\partial^2 v_1}{\partial x\partial y}\right]. \tag{5.5}$$

These four equations (5.2)–(5.5) thus describe stability of the unidirectional flow.

It is now assumed that Tollmien–Schlichting type disturbances [7] are made to the basic flow, with sinusoidal perturbations in the lateral $y$-direction and in the direction $x$ of propagation. Accordingly, the flow variables are expressed in the forms

$$\begin{aligned}
u_1(x, y, z, t) &= \hat{u}_1(z)\exp[i(kx + \gamma y - \omega t)],\\
v_1(x, y, z, t) &= \hat{v}_1(z)\exp[i(kx + \gamma y - \omega t)],\\
w_1(x, y, z, t) &= \hat{w}_1(z)\exp[i(kx + \gamma y - \omega t)],\\
p_1(x, y, z, t) &= \hat{p}_1(z)\exp[i(kx + \gamma y - \omega t)].
\end{aligned} \tag{5.6}$$

At this point, it is appropriate to nondimensionalize the resulting equations, using some typical length scale $H$ and speed scale $S$. For the two-dimensional Poiseuille flow (3.2), the length $H$ corresponds to half the distance between the horizontal plates above and below the fluid, and the speed scale can conveniently be set to $S = GH^2/\mu$ as a measure of the effect of the driving pressure gradient $G$. Thus the base flow in (5.1) in these dimensionless variables would become $U_0(z) = \frac{1}{2}A(1 - z^2)$ over $-1 < z < 1$. This gives rise to the five dimensionless parameters

$$\begin{aligned}
K = kH, \quad \Gamma = \gamma H, \quad \Omega = \omega H/S,\\
\mathrm{Re} = SH\rho/\mu, \quad F = \rho H^2/\tau,
\end{aligned} \tag{5.7}$$

that are pertinent to this system. The first two parameters $K$ and $\Gamma$ are wavenumbers, and are real numbers. The third quantity $\Omega$ is a dimensionless frequency, and is in general complex. If its imaginary part is negative, then the disturbances (5.6) decay with time and so the flow is stable; however, if $\mathrm{Im}\{\Omega\} > 0$ then the flow is unstable. The fourth constant Re in (5.7) is the usual Reynolds number that measures the inverse viscosity $\mu$, and the additional viscosity coefficient $F$ similarly gives a dimensionless measure of the inverse of the second viscosity coefficient $\tau$. In the rheological literature, it often proves convenient to define an alternative dimensionless parameter $Wi = (\tau/\mu)S/H$ known as the Weissenberg number (see Poole [23]). This gives a measure of the ratio of elastic to viscous forces. The parameter $F$ in the system (5.7) can then be expressed as $F = \mathrm{Re}/Wi$. Here, however, the parameter $F$ will be retained, since it gives a pure dimensionless representation of the nonlinear viscosity, independently of the Reynolds number. Finally, a third dimensionless constant should also be introduced, to account for the coefficient $\sigma$ in the equations (5.3)–(5.5);

however, it will be seen that eliminating the pressure term $\hat{p}_1(z)$ in this system also cancels off the parameter $\sigma$, in accordance with the full vorticity equation (4.3).

The forms (5.6) are substituted into the governing equations (5.2)–(5.5), and yield a system of four ordinary differential equations for the four quantities $\hat{u}_1$, $\hat{v}_1$, $\hat{w}_1$ and $\hat{p}_1$. It follows at once from the continuity equation (5.2) that the quantity $\hat{u}_1$ may be eliminated by means of the relation

$$\hat{u}_1 = \frac{i}{K}\frac{d\hat{w}_1}{dz} - \frac{\Gamma}{K}\hat{v}_1. \tag{5.8}$$

The pressure variable $\hat{p}_1$ is then likewise eliminated from the remaining three momentum equations by cross-differentiation. This gives two differential equations for the two variables $\hat{v}_1$ and $\hat{w}_1$. The first equation is

$$
\begin{aligned}
&K(K^2 + \Gamma^2)U_0(z)\hat{v}_1 + iK\Gamma\left[U_0'(z)\hat{w}_1 - U_0(z)\frac{d\hat{w}_1}{dz}\right] \\
&\quad + \frac{1}{\mathrm{Re}}\left[i(K^2 + \Gamma^2)\left(\frac{d^2\hat{v}_1}{dz^2} - (K^2 + \Gamma^2)\hat{v}_1\right) + \Gamma\left(\frac{d^3\hat{w}_1}{dz^3} - (K^2 + \Gamma^2)\frac{d\hat{w}_1}{dz}\right)\right] \\
&\quad + \frac{1}{2F}U_0'(z)\left[iK\Gamma\left(\frac{d^2\hat{w}_1}{dz^2} + (K^2 + \Gamma^2)\hat{w}_1\right) - 2K(K^2 + \Gamma^2)\frac{d\hat{v}_1}{dz}\right] \\
&\quad - \frac{1}{2F}U_0''(z)K\left[i\Gamma\frac{d\hat{w}_1}{dz} + (K^2 + \Gamma^2)\hat{v}_1\right] \\
&= \Omega\left[-i\Gamma\frac{d\hat{w}_1}{dz} + (K^2 + \Gamma^2)\hat{v}_1\right]. \tag{5.9}
\end{aligned}
$$

The second of these differential equations is

$$
\begin{aligned}
&K^2\left[\Gamma U_0(z)\hat{w}_1 + iU_0'(z)\hat{v}_1 + iU_0(z)\frac{d\hat{v}_1}{dz}\right] \\
&\quad + \frac{1}{\mathrm{Re}}K\left[i\Gamma\left(\frac{d^2\hat{w}_1}{dz^2} - (K^2 + \Gamma^2)\hat{w}_1\right) - \left(\frac{d^3\hat{v}_1}{dz^3} - (K^2 + \Gamma^2)\frac{d\hat{v}_1}{dz}\right)\right] \\
&\quad + \frac{1}{2F}U_0'(z)\left[\Gamma(-K^2 + \Gamma^2)\frac{d\hat{w}_1}{dz} + i\Gamma^2(K^2 + \Gamma^2)\hat{v}_1 - \Gamma\frac{d^3\hat{w}_1}{dz^3} - i(\Gamma^2 + 2K^2)\frac{d^2\hat{v}_1}{dz^2}\right] \\
&\quad - \frac{1}{2F}U_0''(z)K^2\left[\Gamma\hat{w}_1 + 3i\frac{d\hat{v}_1}{dz}\right] + \frac{1}{2F}U_0'''(z)\left[i(\Gamma^2 - K^2)\hat{v}_1 + \Gamma\frac{d\hat{w}_1}{dz}\right] \\
&= \Omega K\left[\Gamma\hat{w}_1 + i\frac{d\hat{v}_1}{dz}\right]. \tag{5.10}
\end{aligned}
$$

These two linear equations are the equivalent of the Orr–Sommerfeld equation in classical viscous stability theory [7].

Attention now turns to the specific problem of assessing the stability of the two-dimensional Poiseuille flow solution of (2.4), for which the speed component in the $x$-direction is $U_0(z) = \frac{1}{2}A(1 - z^2)$, as the nondimensionalized equivalent of equation (3.2). There are rigid walls on the planes $z = \pm 1$, and so all three velocity

components must be zero there, according to the usual no-slip boundary condition. In view of the reduced continuity condition (5.8), there are thus six boundary conditions

$$\hat{v}_1 = \hat{w}_1 = \hat{w}_1' = 0 \quad \text{on } z = \pm 1 \tag{5.11}$$

to be satisfied. This problem is solved here using a spectral method.

When analysing the stability of this flow for the regular Navier–Stokes equation by means of its Orr–Sommerfeld equation, Orszag [22] expanded the solution variables as series in Chebyshev polynomials. Here, however, the two solution functions $\hat{v}_1$ and $\hat{w}_1$ are expressed in terms of trigonometric basis functions, in the forms

$$\hat{v}_1 = \sum_{n=1}^{\infty} A_n \sin(n\pi(z+1)),$$

$$\hat{w}_1 = \sum_{n=1}^{\infty} B_n[1 - \cos(n\pi(z+1))]. \tag{5.12}$$

The second of these equations has been chosen so that both the function $\hat{w}_1$ and its first derivative are zero at $z = \pm 1$ in accordance with the boundary conditions (5.11). Notice that these eigenfunction modes (5.12) are symmetric. It is also possible to set up similar expansions involving odd functions, and this has been done in this investigation, but turns out to give essentially identical results; in addition, Orszag [22] points out that, for the regular Orr–Sommerfeld equation, the symmetric modes are the most unstable. Thus, only the symmetric case (5.12) is considered further here.

The solution forms (5.12) are now substituted into the two equations (5.9) and (5.10). The first equation (5.9) is multiplied by the odd basis functions $\sin(k\pi(z+1))$, $k = 1, 2, 3, \ldots$, and integrated over the interval $-1 < z < 1$. Similarly, the second one (5.10) is multiplied by even functions $\cos(k\pi(z+1))$ and integrated. Quadratures involving products of the quadratic function $U_0(z)$ with two trigonometric terms are evaluated exactly, to retain accuracy. The calculations are straightforward, although lengthy. The series (5.12) are truncated to $N$ terms, for numerical purposes, and eventually a large matrix system of linear algebraic equations is obtained, in the form

$$\begin{bmatrix} \mathbf{M}^{(1)} & \mathbf{M}^{(2)} \\ \mathbf{N}^{(1)} & \mathbf{N}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix} = \Omega \begin{bmatrix} (K^2 + \Gamma^2)\mathbf{I} & -i\Gamma\mathbf{Z} \\ iK\mathbf{Z} & -K\Gamma\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix}. \tag{5.13}$$

In this system, each of the quantities $\mathbf{M}^{(1)}$, $\mathbf{M}^{(2)}$, $\mathbf{N}^{(1)}$ and $\mathbf{N}^{(2)}$ is an $N \times N$ constant matrix, with elements derived from the Fourier analysed equations (5.9) and (5.10). The symbol $\mathbf{I}$ represents the $N \times N$ identity matrix and $\mathbf{Z} = \text{diag}(k\pi)$, $k = 1, 2, \ldots, N$, is a diagonal matrix. The symbols $\mathbf{A}$, $\mathbf{B}$ are each vectors of the first $N$ coefficients $A_n$, $B_n$, $n = 1, 2, \ldots, N$, in the representations (5.12).

The system of linear equations (5.13) is a block generalized eigenvalue problem, with eigenvalues $\Omega$. These determine the stability of the flow, as explained above; if any eigenvalue has $\text{Im}\{\Omega\} > 0$ then the flow is unstable. This system (5.13) is solved here using the implementation of the QR generalized eigenvalue routine encoded in the MATLAB function `eig`. The process is fast and accurate, and results with $N = 135$
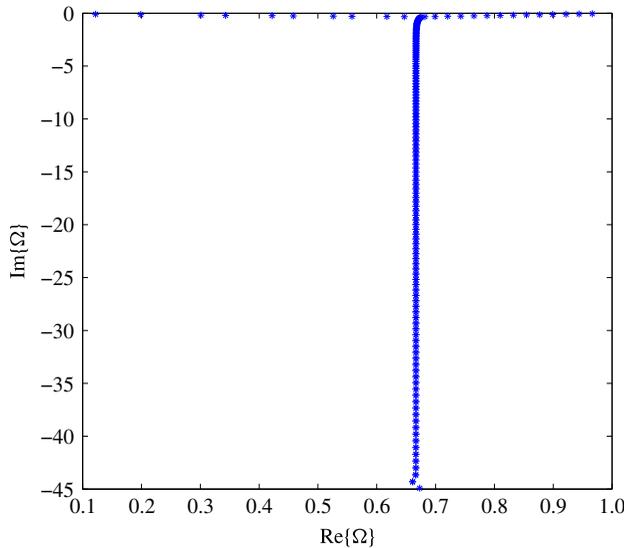
FIGURE 1. Eigenvalue distribution in the complex plane, for the symmetric modes in the pure Navier–Stokes equation. The Reynolds number is Re = 4000 and the pressure gradient creates speed amplitude $A = 2$. Results are shown for both wavenumbers $K = \Gamma = 1$, and there are $N = 135$ modes in each variable.

modes may be obtained in a few minutes of computer time. The equivalent system to (5.13), for the Navier–Stokes equation, has also been derived and solved numerically in the same way, and there is no need to detail that process further here.

Figure 1 shows the eigenvalues in the complex $\Omega$-plane, computed for the Navier–Stokes stability problem with Reynolds number Re = 4000. The base flow is $U_0(z) = \frac{1}{2}A(1 - z^2)$, and the amplitude parameter has been set to $A = 2$ so that the maximum speed at $z = 0$ is $U_0(0) = 1$. The number of Fourier modes in equation (5.12) for each of the two quantities $\hat{v}_1$ and $\hat{w}_1$ has been taken to be $N = 135$, so that (5.13) is a $270 \times 270$ block generalized eigenvalue problem. In this diagram, both wavenumbers have been taken to be $K = \Gamma = 1$. All the eigenvalues have the property Im$\{\Omega\} < 0$, although in some instances the imaginary part of the eigenvalue is small. As a result, this flow is (marginally) stable, as is to be expected with this value Re = 4000 of the Reynolds number.

In Figure 2 two further solutions of the Navier–Stokes equations are also investigated. These have the same Reynolds number Re = 4000 as in Figure 1, but now their wavenumbers have been increased. Thus in Figure 2(a) the wavenumbers are $K = \Gamma = 10$ and in Figure 2(b) the eigenvalues are shown for wavenumbers $K = \Gamma = 50$. In both cases, all the eigenvalues have the property Im$\{\Omega\} < 0$, so that these two solutions are stable; however, as the wavenumbers are increased, the imaginary part of the least stable eigenvalue decreases, so that the flows actually become more stable. This is evident in comparing the eigenvalue distribution for the two situations in Figure 2 with that in Figure 1.
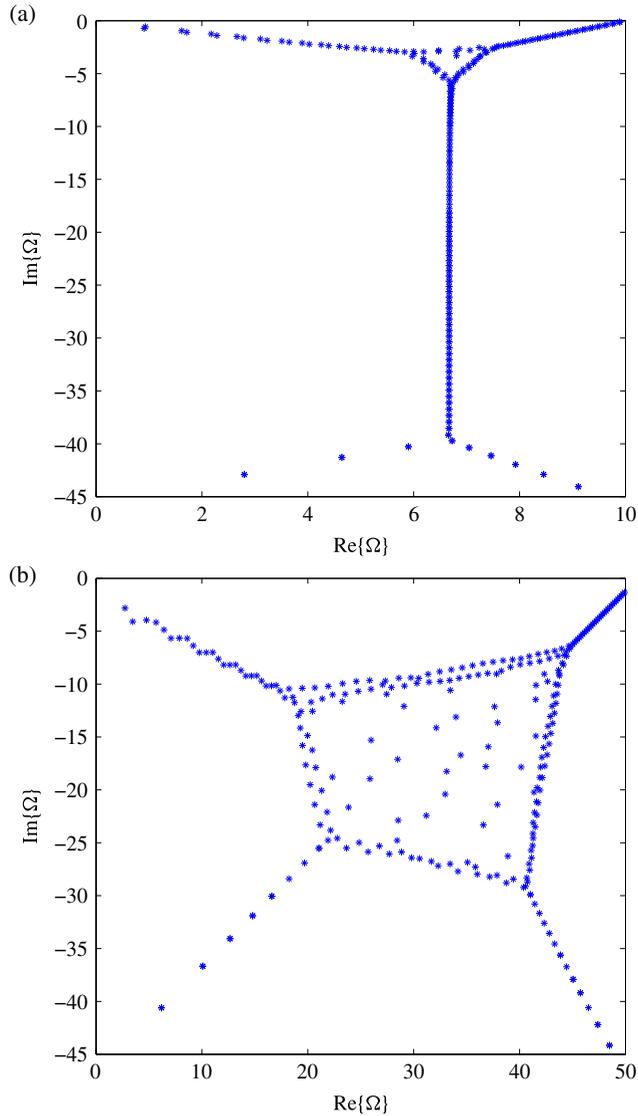
FIGURE 2. Eigenvalue distribution in the complex plane, for the symmetric modes in the pure Navier–Stokes equation. The Reynolds number is Re = 4000 and the pressure gradient creates speed amplitude $A = 2$. Results are shown for (a) wavenumbers $K = \Gamma = 10$ and (b) wavenumbers $K = \Gamma = 50$. There are $N = 135$ modes in each variable.

The situation corresponding to Figure 1 is examined in Figure 3, for the Reiner–Rivlin type equation (2.4). The Reynolds number is again Re = 4000, and in both these diagrams the wavenumbers have been set to $K = \Gamma = 1$. For the results in Figure 3(a), the second viscosity parameter has the value $F = 4000$. All the eigenvalues have
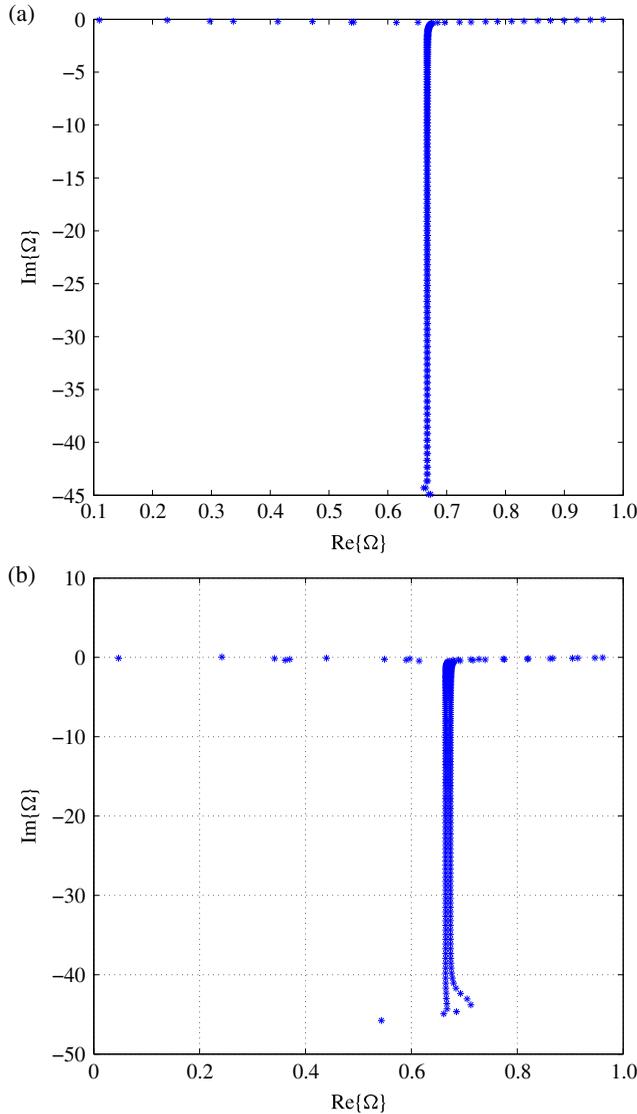
(a)



(b)



FIGURE 3. Eigenvalue distribution in the complex plane, for the symmetric modes in the Reiner–Rivlin type equation. The Reynolds number is Re = 4000, the pressure gradient creates speed amplitude $A = 2$ and the two wavenumbers are $K = \Gamma = 1$. Results are shown for nonlinear viscosity parameter (a) $F = 4000$, and (b) $F = 400$.

$\text{Im}\{\Omega\} < 0$ and so the flow is stable; in fact, there is no appreciable difference between this result and the eigenvalue distribution in Figure 1 obtained for the pure Navier–Stokes equation. In this case, then, the additional nonlinear viscosity terms have no effect on the stability properties of the flow. In Figure 3(b), however, the second
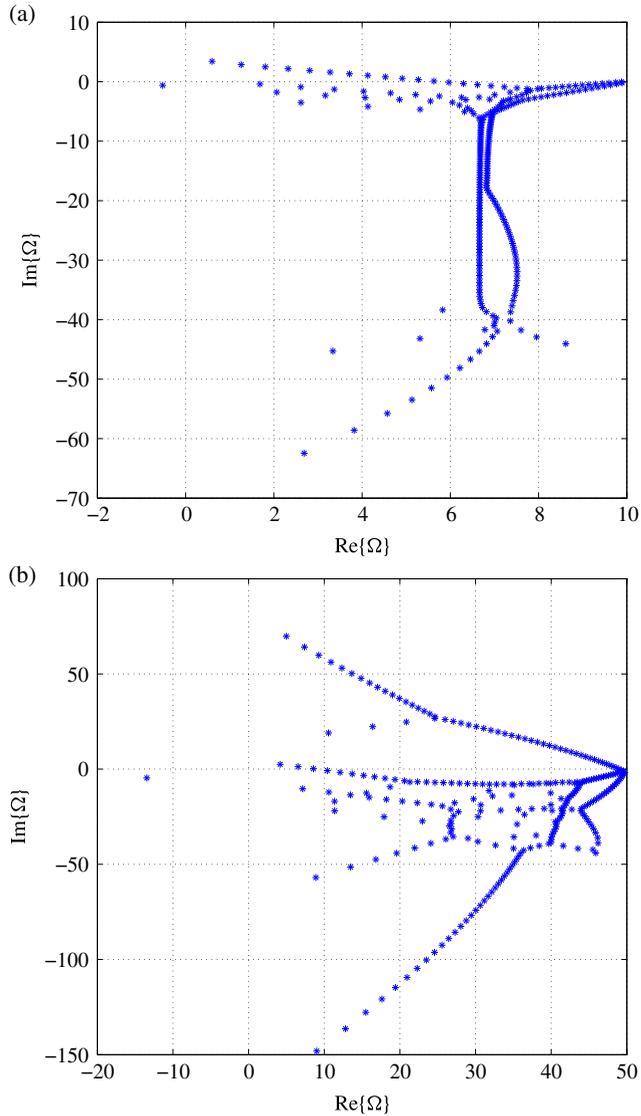
FIGURE 4. Eigenvalue distribution in the complex plane, for the symmetric modes in the Reiner–Rivlin type equation. The Reynolds number is Re = 4000, the pressure gradient creates speed amplitude $A = 2$ and the nonlinear viscosity parameter is $F = 400$. Results are shown for wavenumbers (a) $K = \Gamma = 10$, and (b) $K = \Gamma = 50$.

(inverse) viscosity coefficient has been reduced to $F = 400$. The consequence is that the flow is now (marginally) unstable, since there is a single eigenvalue with positive imaginary part at about $\Omega = 0.2422 + 0.0563i$. This is clearly a case in which the flow is destabilized by the presence of the additional nonlinear viscosity terms.

The effect of wavenumber on these flows is investigated in Figure 4. Here, the same two sets of wavenumbers $K = \Gamma = 10$ and $K = \Gamma = 50$ as in Figure 2 have been used, with the same Reynolds number Re = 4000. However, in Figure 4 the Reiner–Rivlin equation (2.4) is involved, with second viscosity parameter $F = 400$. Both these flows are now unstable, since both of them clearly have eigenvalues with Im$\{\Omega\} > 0$. Furthermore, the imaginary parts of these eigenvalues actually increase as $K$ and $\Gamma$ are increased; thus these solutions become more unstable as the wavenumber is made larger. This is precisely the opposite effect to that encountered with the pure Navier–Stokes equation, as illustrated in Figure 2.

## 6. Discussion and conclusions

This paper takes a slightly unconventional, philosophical approach to the question of turbulence, since it advances the hypothesis that turbulent behaviour might be a manifestation of non-Newtonian material nonlinearity in the fluid. The consequences of this hypothesis for classical-type transition-flow stability analyses have been investigated. It has been found here that a general nonlinear relation between the stress tensor and the strain-rate tensor can indeed cause simple viscometric-type flows to become unstable at lower Reynolds numbers than suggested by pure Navier–Stokes theory, and this is in accordance with calculations [37] in the rheological literature. Furthermore, such a nonlinear relationship between stress and strain rate may cause higher wavenumber disturbances to become *more* unstable, in direct contrast to the results from Navier–Stokes theory. The simplest nonlinear relation between stress and strain rate is assumed here, and is purely algebraic, so that time derivatives of the rate of strain tensor (as in [25]) are not considered.

A second interesting consequence of this nonlinear Reiner–Rivlin type model in equation (2.4) is summarized in Theorem 3.1, which states that the equation produces flow patterns that are identical to those predicted by the Navier–Stokes equations, *except in three-dimensional flow*. For the conventional Navier–Stokes equation, "Squire's theorem" [33] shows that two-dimensional perturbations to a flow must be more unstable than a comparable three-dimensional disturbance. Clearly, however, the nonlinear viscosity model in equation (2.4) must be exempt from Squire's theorem, by virtue of Theorem 3.1. Thus, a planar perturbation to a simple flow in a Reiner–Rivlin fluid at a moderately low Reynolds number will be stable, since it reduces in that case to a Navier–Stokes flow; however, a fully three-dimensional small-amplitude disturbance to that same flow may well be unstable, and particularly if high wavenumbers are involved.

As discussed in Section 1, it is customary to assume that turbulence is described by the Navier–Stokes equations, and this is often stated without further discussion. However, these famous equations are predicated on the Newtonian assumption of a linear relationship between stress and strain rate, analogously to Hooke's law in elasticity. While this is appropriate for laminar flows, where local strain rates are small to moderate in size, it may possibly be insufficient for turbulence, precisely

because in that case local strain rates are large. Philosophically, there is thus an inherent contradiction involved in assuming the universal validity of the Navier–Stokes equations in circumstances where local strain rates are high, such as occurs in turbulence. In effect, this contradiction is tacitly acknowledged in practical turbulence models such as *k*-epsilon theory [31] for example, since they involve a momentum equation that looks at least superficially like a type of rheological equation, while nevertheless stating that Navier–Stokes theory still applies.

Of course, the Navier–Stokes equations are nonlinear, and able to produce all the complex structures associated with chaos theory. The question is then whether this chaotic behaviour alone is sufficient to describe turbulence, and this relationship is discussed by McComb [21]. He urges caution about the over-reliance on low-dimensional chaos as a full explanation for turbulence. Indeed, it may be timely to question again the continued use of simple averaging techniques, introduced in 1895 by Osborne Reynolds [27], combined with semi-heuristic models [4] to cope with the problem of closure. Such an approach would not be entertained in analysing a canonical chaotic system such as the Lorenz attractor [12], for example; in that case, while it might be possible to select a closure model that could give an orbit with similar statistical properties to the true solution, it nevertheless seems highly unlikely that the precise intricacies of the true Lorenz attractor would be reproduced faithfully.

If the present hypothesis is true, that turbulence is a result of the nonlinear relation between fluid stress and strain rate, then it follows that Reynolds number alone does not give a complete description of when such behaviour can be expected. This is already known to be true in highly rheological fluids, and Larson [18] shows an example of polymer flow apparently exhibiting turbulence at Reynolds number $10^{-15}$. In this view, then, turbulence as commonly encountered in water is a phenomenon lying on a continuum of behaviours that includes such exotic structures formed in polymers.

This hypothesis also raises many other questions. If it is indeed the case that the transition to turbulence is more complicated than a linearized stability analysis such as in Section 5 would suggest, and large-scale nonlinear structures are also involved [13, 36], then it is of interest to know their detailed behaviour in a system such as equation (2.4). Additionally, since the unstable transition flows for the Reiner–Rivlin type fluid presented in Figure 4 possess many eigenvalues with positive imaginary parts, for which the frequencies do not occur as integer ratios, these eigenvalues might trigger a rapid passage to chaotic behaviour through high-dimensional quasi-periodic orbits. Finally, detailed numerical solutions of (2.4) in practical flow situations are also of interest. These questions await future research.

## Acknowledgements

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