THE ROLE OF THE CONCEPTUAL METAPHOR IN THE DEVELOPMENT OF CHILDREN'S ARITHMETIC

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This paper explores the relationship between the perceptual and the conceptual in the development of children's informal arithmetic. It compares two major theories- Piagetian abstraction and embodied learning – in order to clarify the building of abstract ideas from perceptual, sensory experiences. The arithmetic principles of commutativity and associativity are examined within these two theories. The theory of embodied learning and the conceptual metaphor is considered as a lens for examining children's informal, intuitive arithmetical knowledge.

INTRODUCTION

This paper is presented as an examination of theoretical issues and existing empirical research that explores how children's formal mathematical ideas can be built from informal, intuitive arithmetic. Although research has examined children's invented procedures and the flexibility of the procedures (Carpenter and Moser, 1984; Steinberg, 1985; Kamii, Lewis, and Jones, 1993; Foxman and Beishuizen, 1999) the research into how children develop the ability to use flexible methods is more limited, although Gray and Tall (1994) have associated success in the use of arithmetic flexibility to the notion of a 'procept' where numbers are viewed as both processes and concepts.

Arithmetic principles, such as commutativity and associativity, can play a role in the development of flexible calculation strategies. There is evidence (Groen and Resnick, 1977) that children develop the use of arithmetic principles without instruction. They come to use the arithmetic principles intuitively from their own informal, spontaneous development of arithmetic.

If there is evidence that children develop an intuitive and spontaneous use of principles such as commutativity, how does this happen? This paper intends to examine two theoretical models - Piagetian theory of abstraction and the more recent theory of embodied learning - that both provide a model for the development of the arithmetic principles.

CHILDREN'S IMPLICIT USE OF PRINCIPLES OF ARITHMETIC

There is evidence that prior to or in the absence of direct instruction young children will devise their own procedures that assume mathematical principles. Groen and Resnick's (1977) empirical work with 4- year olds showed that, even though instruction in addition was limited to the 'count-all' strategy with physical objects, many children soon abandoned this more primitive strategy and initiated the 'count-on' strategy. They also found that many of the children spontaneously chose to start with the larger number. For example, given the problem 2 + 7, the children swapped the numbers to 7 + 2 in order to make the 'count on' more efficient. This 'count-on from the larger' strategy assumes commutativity in that $2 + 7 = 7 + 2$ or, more formally $a + b = b + a$. Although still relying on a counting procedure there is an implicit use of an arithmetic principle in order to use a more economical strategy.

Further to the more efficient count-on strategy children may use known facts in an innovative way as they invent their own arithmetic procedures. Beishuizen, Van Putten and Van Mulken (1997) and Fuson (1992) have identified two main types of invented procedures. One is termed 'splitting' numbers where tens and units are dealt with separately (23 + 4: a child may add the 3 + 4 and then add to the 20). Another is termed 'complete' number where one number is kept complete (24 + 7: a child may keep the 24 complete but split the 7 into 6 and 1). Such 'splitting' number or 'complete' number procedures assume associativity in that $(20 + 3) + 4 = 20 + (3 + 4)$ or formally $(a + b) + c = a + (b + c)$.

Children who invent their own procedures would appear to have an intuitive understanding of arithmetic principles such as commutativity and associativity. It is possible that children come to assume that arithmetic operations are commutative as they realise the principle of order irrelevance (Gelman and Gallistel, 1978) and apply this assumption to addition.

> "Addition in the child's view, involves uniting disjoint sets and then counting the elements of the resulting set. According to the order irrelevance principle it does not matter whether in counting the union you first count the elements of one set and then the elements from the other or vice versa" (p. 191).

When extended to three sets, associativity is also used intuitively.

THEORETICAL PERSPECTIVES

The spontaneous development of the principles of commutativity and associativity would suggest that children bring intuitive, informal arithmetic to the classroom. Descartes' notion of intuition is that of certain and evident knowledge (Lakoff and Johnson, 1999) where we cannot help but see what is before us. 'Knowing is seeing' is a tenet of Lakoff and Johnson's view of embodied learning. Phrases such as 'I see what you mean' or 'Let's see what is in the box' are used to convey knowledge of what has been said or knowledge of what is in the box. From an embodied learning viewpoint perception, as a sense-impression from the external world, is seen as a source domain for knowledge. Conceptualisation of abstract ideas can be reasoned about from domains of experience, of which many are sensory-motor. The cognitive mechanism for such conceptualisation is the conceptual metaphor. The conceptual metaphor is not merely a figure of speech but a matter of thought (Lakoff, 1980). It is the mechanism by which the abstract is comprehended in terms of concrete, everyday, sensory-motor experiences such as 'in', 'next' or 'movement'.

Lakoff and Nunez's (2000) analysis of mathematical ideas suggests an elaboration of everyday commonplace experiences such as object collection and object construction onto the abstract world of number. Mathematical reasoning is seen as a product of bodies and brains. The notion of embodied learning presents a mechanism to work up from sensory experiences to abstract concepts. Through metaphorical projection abstract concepts are brought into being from the sensory, figurative world (Johnson, 1987).

In the Piagetian viewpoint concepts of number and arithmetic are not seen to be developed through sensory-motor experiences but through reflective or pseudo-empirical abstraction. Whereas empirical abstraction described the unconscious abstractions from the sensory-motor elements and the observable properties of objects themselves (Piaget, 2001), reflective abstraction described an operation on a mental entity that becomes in turn an object for reflection at the next level, allowing for further mental operations (Gray, Pinto, Pitta, and Tall, 1999). Although a two-stage hierarchical process reflective abstraction does not draw its information from the sensory, physical experiences of empirical abstraction but from the coordination of the objects. Empirical abstraction has no parallel hierarchy. That is, there is no empirical abstraction from the results of previous empirical abstraction (Piaget, 2001). There is no projection from perceptual knowledge and so perceptual knowledge cannot be the source of new constructions.

ABSTRACTION AND ARITHMETIC PRINCIPLES

In Piagetian terms abstraction in the development of number and arithmetic is non-empirical. The notion of number is not supplied by the senses so there is a need to attend to non-perceptual properties of the objects. Abstraction of this form is termed pseudo-empirical. It draws its information from apprehending the properties that are presented by an object but where the properties were introduced by previous actions. The focus is on the actions of the objects and the properties of those actions. The child may be 'leaning' on the perceivable results but the perceived properties have been introduced by the child's actions. Such an abstraction entails a level of reflection.

The spontaneous development of the arithmetic principles, such as commutativity, would occur as a form of pseudo-empirical abstraction. The source is drawn from the coordination of the actions of counting and manipulating the objects. The coordination of objects may impress on a child that there is a reason for a particular result, a 'quasi-necessity' (Piaget, 2001), where a child is certain of an event even though the child may not understand the reason for it. The child gains the impression or assumption of commutativity. In Piagetian theory the sensory, experiential world has no direct relationship with the child's assumption of commutativity.

CONCEPTUAL METAPHOR AND ARITHMETIC PRINCIPLES

The notion of embodied learning provides a model where perceptual, sensory-motor experiences are part of the formation of concepts. Abstract ideas can be conceptualised and reasoned about from domains of experience that are mostly sensory-motor (Lakoff and Johnson, 1999). In understanding an idea, we may talk about 'grasping an object' (a sensory-motor experience) and if we fail to understand an idea we talk about it 'going over our heads' (a sensory-motor experience). Such sensory-motor structuring is apparent in the sense of quantity where we say that 'more is up'. Here 'more' is conceptualised in terms of the sensory-motor experience of verticality, which may have derived from the filling of a glass of water.

Lakoff and Nunez (2000) have provided a model for the development of arithmetic principles from perceptual systems. From the properties of object collections we can determine equal results through different operations in the construction of the collections and see that the same collection results from any order. More specifically, the knowledge that combining object collections A and B in the physical world give the same result as combining B to A can be mapped onto the number world (p.54). This would be similar for three sets.

Other everyday experiences show us that there are various ways to get the same results. Lakoff and Nunez gave the example of shopping for an item by going to the shops, by mail catalogue or over the Internet. These are all different processes that result in the purchase of an item. Such knowledge is represented as an Equivalent Result Frame (Lakoff and Nunez, 2000, p.87). The conflation of these metaphors, Arithmetic is Object Collection and Arithmetic is Object Construction, with the Equivalent Result Frame would explain the emergence of commutativity and associativity in children's arithmetic. Here the abstract reasoning of commutativity or associativity is based on the perceptual experiences of seeing the identical result of the combinations or the different processes in everyday life.

DISCUSSION

Both theoretical perspectives have provided explanations for the spontaneous development of arithmetic principles that are used in flexible calculation strategies. The Piagetian viewpoint would seem to provide a model for examining the perceptual separately from the conceptual. The embodied learning perspective would seem to provide a model where the conceptual can be built from perceptual, sensory experiences.

Baroody and Ginsburg's (1987) empirical research provided an example of a boy, Case, who appeared to be uncertain of applying commutativity to addition procedures. When asked if commuted pairs such as 6 + 2 and 2 + 6 would add up to the same thing or something different, Case's response was that the pairs were '*almost* the same but different'. When asked to add 2 + 7 and 7 + 2 Case seemed uncertain whether the commuted pairs were equivalent or not and carried out counting procedures with both pairs to check. Why would he say '*almost* the same but different'? We do not know for certain why Case responded with the statement but it is tantalising to speculate.

One speculation may be that Case is focusing on the perceptual attributes of the object collections. After all a collection of 2 objects of one type, say colour, and 7 of another would appear as a different collection to one of 7 and 2. The arrangement of the objects would give a different pattern. Hence focus on the 'rich image of the objects' would not suggest that the commuted pairs are the same. Pitta's (1998) empirical studies of young children and counting cubes suggested that the more able children attended to mathematical qualities, such as the notion of five when asked to say what was important about the set of objects. The children who did not use efficient strategies would focus on the concrete experiences such as pattern or colour. Thus it is possible that some children did not know what was relevant to focus on. In the same way that Tall (2004) proposed the need to focus on the non-perceptual attributes of objects, maybe Case has not rejected the 'rich image of detail' of the objects in order to focus on the structural relations. This speculation would support the Piagetian notion of reflected abstraction that draws on the non-perceptual attributes. The focus is on the coordination of the objects and not the objects themselves.

Take the situation where you have 2 sweets and are given 7, this would be a very different situation to having 7 sweets and being given 2. Even the action of counting out 2 sweets and counting out 7 is different to counting out 7 and the counting out 2. So if the child focused on the 'rich image' of the actions the commuted pairs may not be seen as equivalent.

In mapping from the object collection metaphor of the physical world to the world of numbers the physical attributes would not seem to be helpful in making sense of the situation and seeing the equivalence. Sfard (1994) commented on the implausibility of the claim that metaphorical projection from the perceptual to the abstract could be a simple correspondence between a sensory experience and an abstract concept in a similarity relation. Sfard has interpreted the embodied notion of conceptual metaphor as non-comparative. She saw the conceptual metaphor as

> "… a mental construction which plays a constitutive role in structuring our experience and in shaping our imagination and reasoning. In other words, rather than being a product of a comparison between two existing things or ideas, metaphor, as conceived by Lakoff and Johnson, is what *brings abstract concepts into being*" (p. 46)

Sfard continued to explore the notion of the embodied schemata as those originally built to put order into our physical experience, which are "'borrowed' to give shape, structure and meaning to our imagination" (p. 47). She proposed that this view of embodied learning and the conceptual metaphor does acknowledge the abstract reasoning from the physical world or 'figurative projection'.

But how might the embodied learning perspective explain Case's uncertainty with the commuted pairs? If I am allowed to take a comment from one of Sfard's interviewees, it may shed some light.

> "It is only when you are perfectly certain, without having to check, that things might be exactly the way they are. It's like in the case of an intimate familiarity with a person. With such a person you often know what he is going to do without having to ask … The (abstract) things have a life of their own but if you understand them, you make predictions and you are pretty sure that you will eventually find whatever you foresaw… The intimacy is exactly what I had in mind: you know what is to happen without making any formal steps…" (p. 49)

This response reflects Johnson's (1987) view of understanding, that it is not just a matter of reflection on pre-existing knowledge but as the "way we experience our world as a comprehensible reality" (p. 102). If, as Sfard suggested, "experiential comprehension gives people an ability to anticipate behaviours of material objects without reflection" (p. 49), a further speculation might be that Case has not yet been able to use experiences to support anticipation and certainty of the result. The conceptual metaphors of everyday experiences of equivalent results may help him feel familiar with initutive ideas such as commutativity and to make sense of the abstract mathematical notion. A young child's understanding of commutativity may not solely be through operational reasoning and reflection on the process but also through analogical reasoning on the equivalent results of the process as an ontological object, a familiar known experience.

DEVELOPMENT OF AN EMPIRICAL STUDY

This theoretical discussion proposes that the conceptual metaphor can have a role in the construction of children's arithmetic and acknowledges the possibility that mathematical concepts such as number can be built from bodily actions and perceptions. An empirical study would allow further substantiation of such a proposal. In the area of neuroscience empirical evidence would suggest that embodied learning and the notion of conceptual metaphor does play a role in the development of mathematical ideas. As Rogers and Caines (2007) proposed, mental processes can be said to exist by virtue of neural processes where human ideas such as number have their origins in bodily perceptions. Metabolic brain imaging techniques provide evidence that intuitive ideas are part of a functional web that connects primary sensory and motor areas.

But how could a methodology be developed to investigate the role of the conceptual metaphor in children's learning in arithmetic? The development of an empirical study would suggest methodological difficulties, one of these difficulties being how to investigate implicit, intuitive knowledge in young children's solving of numerical problems.

In Sfard's study she had asked the question 'What happens in your mind when you feel that you have understood a piece of mathematics?' to research mathematicians. Responses suggested a notion of familiarity with the mathematics. As mathematicians it is possible that they were able to see 'right to the ground'. In other words they were explicitly aware of the intuitive sense of the mathematics. A brief investigation with prospective primary teachers did not provide the same responses of familiarity. However some of the prospective primary teachers did refer to 'cogs in the brain', 'ideas clicking into place', 'pieces of jigsaw fitting into place' or even 'a flick of a switch'. These in themselves are metaphors for connections so even if they do not mention the notion of familiarity they may be referring to the sense that the mathematical idea is becoming part of a functional web. Whether this question would elicit such responses from children has not been tested.

A further methodology would be to observe children's overt strategies as they carry out trials to solve addition problems and how these relate to their understanding of commutativity or associativity as in Canobi, Reeve and Pattison's (2002) study. Canobi et al's study provided further evidence of the relationship between children's intuitive use of the arithmetic principles and their development of flexible strategies. Children were asked to justify the correct responses and it would seem that the children referred to the equivalence of the results but this is not explored fully. A further examination into the children's autonomous ideas of equivalence as a structural commonality could be pursued.

Other studies have provided evidence that children's analogous reasoning enables them to see the structural commonalities in multiplication and division word problems (English, 1997). It can also be seen that young children may focus on the actions and relationships within different numerical problems. A child may solve a problem through direct modelling where the strategies used reflect the specific actions of the problem and, as such, interpret each problem as a new, individual one (Carpenter, Fennema, Franke, Levi, & Empson, 1999). Young children may not see the 'common thread' or structural commonality that ties the direct modelling strategies together. As children progress to more advanced counting strategies it is considered that they begin to see the common thread. In the same way that English demonstrated that analogous reasoning helped children to see the structural commonalities so it is possible that analogous reasoning enables children to see the 'common thread' or equivalence different contextual problems and even in trials related to commutativity and associativity. Hence as each trial is presented to the children as a problem with different actions and relationships analogy plays a role in allowing them to see the structural commonality of equivalence.

The combined methodologies of observing overt calculation strategies and investigating analogous reasoning could be seen to enable the determination of a relationship between analogy and the development of flexible calculation strategies. A methodology is still needed to determine how the intuitive knowledge is arrived at and to identify the mechanism that allows the analogous reasoning to take place. If conceptual metaphors are involved, how are they processed?

CONCLUSION

The conceptual metaphor and embodied learning perspective may help to explore how sensory experience is built on and provide a way of exploring the relationship between children's informal, intuitive knowledge and the formal knowledge of the classroom. Sensory experiences with object collections could be built on to develop the intuitive knowledge of the arithmetic principles that in turn could support the development of flexible strategies.

In exploring the example of commutativity it is possible to see on one hand the development as a non-empirical abstraction based on reflection of the child's actions on objects but the dissociation from the perceptual does not provide the opportunity to explore the relationship with children's sensory experiences. On the other hand, exploring commutativity from an embodied learning perspective provides the possibility that the abstract world of number can be built from perceptual, sensory experiences.

The issue of divergence in children's use of arithmetic remains a concern for educationalists. It has been suggested that low attaining children rely on procedural counting strategies whereas more able mathematicians recognise the economy of flexible strategies (Baroody and Ginsburg, 1986; Gray, 1991). Children who adhere to procedural counting strategies may find it more difficult to learn flexibility in arithmetic procedures, even with instruction (Murphy, 2004). As further instruction in arithmetic procedures takes place in school, it is possible that mathematics may become a subject that makes little sense to these children. It would seem worthwhile to examine how children make the mental leaps that allow them to understand the mathematics.

As yet little is known how metaphors are processed (Gentner, Holyoak, and Kokinov, 2001) but conceptual metaphors may provide a lens to investigate children's development in arithmetic. The notion of embodied learning could help examine how children build an abstract notion of number and develop an implicit use of principles that informs their arithmetic from informal, sensory experience. The examination of children's development in arithmetic in terms of conceptual metaphors has inherent methodological difficulties in determining young children's explicit awareness of something that is implicit and intuitive. Other studies have paved the way in indicating the relationship between arithmetic principles and flexible strategies and also in children's progression from the 'rich detail' of informal arithmetic and direct modelling to the more abstract counting strategies that rely on analogical awareness of structural similarities. A review of observation or interview techniques is needed to investigate the mechanism that is happening and to determine the role of the conceptual metaphor.

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