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WHAT IS WRONG WITH CANTOR'S DIAGONAL ARGUMENT?

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1. *Introduction*

As a long-time university teacher of formal logic and philosophy of mathematics, the first author has come across a number of students over the years who have cast some doubt on the validity of Cantor's Diagonal Argument.[1] They usually express some amazement about the conclusion of Cantor's Argument, viz, Cantor's Theorem, that there are non-denumerable sets, that is, infinite sets that are of a higher cardinality than the set of natural numbers. Whilst the idea of an infinite set sounds quite plausible to them, they cannot readily countenance the idea that there is more than one level or type of infinity. Unwittingly, I have always given the standard response that the conclusion is inescapable, given that, firstly, the cardinality of a set is determined by one-one correspondence between it and a standard set of given cardinality and, by Cantor's Diagonal Argument, the power set of the natural numbers cannot be put in one-one correspondence with the set of natural numbers. The power set of the natural numbers is thereby such a non-denumerable set. A similar argument works for the set of real numbers, expressed as decimal expansions.

However, students often have pre-theoretical intuitions about a discipline they are studying and can sometimes come up with an idea which may not be in the theoretical ambit of a staff member. A perfect example of this is that, over the years, many a student has been critical of classical logic, usually

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[1] I wish to thank Zach Weber, a post-graduate student at the University of Melbourne, supervised by Graham Priest, for raising this issue with me as recently as November, 2005. It was the ensuing discussion that triggered the writing of this paper.

criticizing the truth-table of the material '$\supset$' or the lack of relevance of conclusions to premises. One only has to see the motivation and development of relevant logics, which pursue these concerns, to realize that there is an important point to the student's persistence in this case. (We will come back to this in section 2 below.) What we aim to show in this paper is that there is also an important point to the student's concerns about Cantor's Diagonal Argument, thus making amends to these students.

But, what could be wrong with Cantor's Argument? It must be something to do with the treatment of infinity. Initially, one would treat infinity as something that can be approached through ever larger finite numbers, as would happen in the process of establishing a limit of a sequence of numbers. In this context, recursive processes, functions and sets would have initial appeal. We will see later how this fits in.

Focussing back on the above two concerns, we can see that they are not unrelated. The concern about material implication and its lack of relevance leads us naturally to consider mathematical structures in the light of recent work on relevant logic, and especially weak relevant logics without the Law of Excluded Middle (see [RLR1 & 2]). As we shall see, such relevant logics provide a weaker base than classical logic and so are more capable of showing up the shortcomings of Cantor's Argument.

To do all this, we first consider the entailment logic MC, based on meaning containment, which contains neither the Law of Excluded Middle (LEM) nor the Disjunctive Syllogism (DS) (see Brady [1996] and [UL]). We then argue that the DS may be assumed at least on the same basis as the assumption of the LEM, which is, we argue, justified over a finite domain or for a recursive property over an infinite domain, or however the LEM might be subsequently proved. In the recursive case, use is made of Mathematical Induction. We then show that an instance of the LEM is instrumental in the proof of Cantor's Theorem, and we then argue that this is based on a more general form than one can reasonably justify, i.e. it is not one of the above justified assumptions. Finally, we briefly consider the impact of our approach on arithmetic and naive set theory, and compare it with intuitionist mathematics and very briefly with recursive mathematics.

Our 'Four Basic Logical Issues' paper [200x], would provide useful background, the current paper being an application of the some of the ideas in it, though readable independently of it. Further background can be provided by the first author's earlier conference paper 'Entailment Logic – A Blueprint' [2007].

2. *The Entailment Logic MC*

Deductive logic is about the deduction of conclusions from premises, where the conclusions follow as a matter of certainty. As in Brady [2007], this certainty is due to the meaning relationships between a conclusion and its premises, i.e. the premises embody some concepts, from which the conclusion follows as a further property of these concepts and only these. In this light, it is reasonable to introduce the inference connective '$\rightarrow$', to be understood as meaning containment, as detailed in Brady [1996] and [UL]. Also, the inference '$\Rightarrow$', used to relate the conclusion to the premises of an argument, is not only truth-preserving, but is also driven by meaning containment. This can indeed be seen by examining the rules of the logic QMC, given below.[2]

So, we introduce below such an entailment logic MC, the entailment being represented by the above '$\rightarrow$'. A particular feature of MC is that each of the other connectives is dependent for its properties on this entailment '$\rightarrow$', in that theorems involving them can be built up from entailment theorems, making entailment theorems the focus of the logic. This follows from the metacompleteness property for MC, as can be seen from Slaney [1984] and [1987]. The metavaluations, for which metacompleteness with respect to the logic holds, are determined inductively for all formulae constructed from entailment theorems. What also follows from the metacompleteness of MC is the important Priming Property: if $\vdash$ A $\vee$ B then $\vdash$ A or $\vdash$ B, an intuitive property that depends in this case on the A or the B being built up from entailment theorems.

MC (previously called DJ^d) is also a depth relevant logic (as can be seen in Brady [1984]), and can be used to solve the set-theoretic and semantic paradoxes (as can be seen in Brady [2000] and, in detail, in [UL]).

We now present the axiomatization of MC and its quantificational extension QMC. We use the system QMC, which is MCQ without the quantificational distribution principles, $\forall x(A \vee B) \rightarrow A \vee \forall xB$ and $A \& \exists xB \rightarrow \exists x(A \& B)$. The term 'QMC' is derived from Mares and Goldblatt's QR in their [2006], which is the relevant logic RQ, also without these distribution principles. We will explain why these principles are dropped in section 9.

MC.
Primitives.
$\sim$, &, $\vee$, $\rightarrow$ (connectives).

[2] Briefly, Rules 1 and 3 exhibit applied meaning containment, whilst the comma of Rule 2 is conjunctively interpreted. For Quantified Rule 1, the premise A must have an unconstrained free variable, upon which the generalisation takes place. So, the universal quantifier exhibits what is already there in the free variable.

p, q, r, ... (sentential variables).
Axioms.
1. $A \to A$.
2. $A\&B \to A$.
3. $A\&B \to B$.
4. $(A \to B)\&(A \to C) \to .A \to B\&C$.
5. $A \to A \vee B$.
6. $B \to A \vee B$.
7. $(A \to C)\&(B \to C) \to .A \vee B \to C$.
8. $A\&(B \vee C) \to (A\&B) \vee (A\&C)$.
9. $\sim\sim A \to A$.
10. $A \to \sim B \to .B \to \sim A$.
11. $(A \to B)\&(B \to C) \to .A \to C$.
Rules.
1. $A, A \to B \Rightarrow B$.
2. $A, B \Rightarrow A\&B$.
3. $A \to B, C \to D \Rightarrow B \to C \to .A \to D$.
Meta-Rule.
1. If $A \Rightarrow B$ then $C \vee A \Rightarrow C \vee B$.

QMC.
Additional Primitives.
$\forall$, $\exists$ (quantifiers).
a, b, c, ... (free individual variables).
x, y, z, ... (bound individual variables).
f, g, h, ... (predicate variables).
Quantificational Axioms.
1. $\forall xA \to A^a/_x$.
2. $\forall x(A \to B) \to .A \to \forall xB$.
3. $A^a/_x \to \exists xA$.
4. $\forall x(A \to B) \to .\exists xA \to B$.
[Note that, in distinguishing free and bound variables, x can only occur bound in the 'A' of Ax2 and in the 'B' of Ax4.]
Quantificational Rule.
1. $A^a/_x \to \forall xA$, where a does not occur in A.
[This rule generalises on the free variable a in $A^a/_x$, with occurrences in exactly the same places as those of the bound variable x in A.]
Quantificational Meta-Rule.
1. If $A^a/_x \Rightarrow B^a/_x$ then $\exists xA \Rightarrow \exists xB$, where a does not occur in A and B.
The meta-rules MR1 and QMR1 carry the proviso that, in the derivations $A \Rightarrow B$ and $A^a/_x \Rightarrow B^a/_x$, the rule QR1 does not generalise on any free variable in A and $A^a/_x$, respectively.

3. *The Lack of the DS and the LEM*

As noted in [UL], pp. 39–41, the logic MC, there called DJ^d, contains neither the Disjunctive Syllogism (DS) nor the Law of Excluded Middle (LEM). The DS ($\sim$A, A $\vee$ B $\Rightarrow$ B), from which Ex Falso Quodlibet (A, $\sim$A $\Rightarrow$ B) easily follows, cannot be a rule of MC because, as in Ch. 6 of [UL], the model MD of the dialectical (inconsistent) set theory DST, based on an extension of the logic DJ^d, is non-trivial, as shown on p. 245 of [UL]. Also, the LEM (A $\vee$ $\sim$A) cannot be a theorem as the Priming Property (see 2) would then fail. Further, the LEM can be used to prove paradoxes such as the Liar, in the form of inconsistencies, but the overarching predicate theory PT, based on the higher-order predicate logic DJ^dHQ is simply consistent, as shown in Ch. 8 of [UL].

Further, as stated on p. 39 of [UL], 'the LEM and the DS ... only involve negation and disjunction, and do not involve the entailment connective', making them 'independent of any considerations to do with meaning containment'. They would have to be justified through the meanings of negation and disjunction. Disjunction is much better understood than negation, and classical negation as a universally applicable negation is hard to justify in a non-circular fashion, as argued in Ch. 1.7 of [UL]. Indeed, it is very doubtful whether there are general logical principles to support either the LEM or the DS and, as also argued in [UL], there are likely examples of each of these principles failing, as well as the many obvious examples of each holding. (We pick this up later in this section.)

In [UL], it is assumed as a result of this, that both of these principles hold for so-called classical sentences[3] and 2-sorted systems are set up, one sort for classical sentences, and one sort for sentences in general where neither principle is assumed. The book goes on to argue that the major mathematical theories, including set theory and arithmetic involve only classical sentences, thus allowing whole classical mathematical theories to be transcribed within the classical sort.

However, as explained in Brady and Rush [200x], there is a general problem justifying the classical sort in that the domain of classicality is hard to define in precise terms, and a single sort is neater and philosophically easier to justify. This paper, together with Brady [2007], suggests metacompleteness and rejection as techniques to prove the DS and the LEM within a single sort. However, as will be explained below, we will derive them where we can and we may assume them outright in other appropriate circumstances. With

[3] The reason we can talk of classicality here is that all the tautologies of classical logic, expressed in the connectives $\sim$, & and $\vee$, can be proved in the logic MC from the LEM, using normal-forming operations, and the detachment rule of classical logic is easily obtained from the DS.

its lower focus on classicality, the single sort will also deal with Richard Sylvan's voiced concern to the first author that I have compromised too much with the classicists.

As we mentioned at the beginning of 2, logic is based on deduction, and should focus on proof theory, and so any semantics ought to be able to characterize proof-theoretic features. A suitable concept of truth for this would be establishable truth, as suggested in Brady [2007]. Being a relatively weak relevant logic without the LEM, the logic MC is likely to turn out to be constructive in some sense, having the lack of the LEM in common with intuitionist logic, but there are differences here in the inferences and in the negation. The important thing in common with intuitionism is the need to derive the unnegated and negated forms of a proposition, which then enables truth and falsity to properly overlap and not be exhaustive. We will briefly compare these logics in section 10 at the end of the paper.

Deductive systems prove results about concepts captured by the axioms or assumptions associated with these concepts. It is quite conceivable that such a system may be simply inconsistent or that it may be $\sim$-incomplete. The best known example of simple inconsistency is the classically based naive set theory, initially introduced by Cantor. There are many examples of concepts that seem to be consistent with each other to start with, but turn out, upon some development of the combined theory of such concepts, to be inconsistent. Examples of such inconsistency are to be found in Priest [2001], pp. 67–8 and pp. 125–6. The best known example of $\sim$-incompleteness, indeed incompletability, is classically based Peano Arithmetic, due to Gödel's Incompleteness Theorem. Also, it would not be hard to find concepts about which a question could be asked without a yes-or-no answer, due to incomplete specification of the concept or concepts. Thus, a 4-valued logic, embracing independent truth and falsity, should form the basis of a natural semantics for deductive systems.

So, the failure of both the DS and the LEM is quite appropriate, certainly for logics such as MC. Indeed, it seems very odd, as pointed out in Brady [2007], for classical logic to include the DS and the LEM, only to find that classical theories can be simply inconsistent or $\sim$-incomplete, respectively. Classical proof theory cannot even ensure the Boolean negation upon which the logic is based. Recall, though, that MC is metacomplete and so has the Priming Property, which in turn ensures that there is a closer connection between the DS and simple consistency and between the LEM and $\sim$-completeness.[4]

[4] The relationships are as follows, for metacomplete systems based on MC:
If the DS holds (as an admissible rule) then the system is simply consistent, provided the system is non-trivial.
If the system is simply consistent then the DS holds, again as an admissible rule.

We further these ideas in the next section, gaining some support from a specific naturalistic perspective for our claim that the LEM is contingent.

4. *The Contingency of the LEM*

It is important to note that one does not have to be a constructivist or have an intuitionist's bent in order to doubt that the LEM is universally applicable. Indeed, quite beside the main conceptually motivated argument presented here (which itself emanates from a basic focus on the derivation of conclusions from premises and has no particular constructivist or intuitionist context for that focus), there are a number of independently motivated arguments supporting the notion that the LEM is, in fact, contingent.

One of the most compelling philosophical arguments begins simply by noting the ubiquity of examples wherein the LEM can be seen to fail: of particular interest to the universal approach taken here are those at the limits of human possibility.[5] An especially compelling argument utilizing such examples is that given within the context of a naturalistic philosophy, of the sort sketched in Maddy's [2002]. Since such a context may also serve to explain something of the general role and nature of MC itself, it is worth briefly exploring Maddy's position further.

The (radically strengthened) ideal[6] behind the naturalistic philosophy of logic Maddy puts forward is that of logic as a neat intersection between the internal and the external, wherein logical laws are as much a feature of an external/objective reality as they are an inherent feature of our own make up. Thus, the apriority or certainty of the logical laws comes from both their conceptual necessity and from the general structure of reality itself. This is the ideal. The method (roughly) is transcendental insofar as the notion of conceptual necessity corresponds to the notion of the general pre-conditions of conceptual thought itself, and empirical insofar as the logical laws are (very generally) empirically veridical.

Thus a naturalistic ideal can be viewed as just one small section of a universal ideal. As we have already indicated, a universal approach focuses on human possibility — on deduction per se, and as such is concerned less with the general structure of reality than with the conceptual structure of deduction itself. External, empirically veridical reality is one of the areas to

If the LEM is a theorem then the system is $\sim$-complete.
If the system is $\sim$-complete then the LEM is derivable through either of its disjuncts.

[5] These are spelled out in Brady and Rush [200x].

[6] This is one of the ideals it can serve, at any rate.

which deduction can be applied, but it applies equally to fiction and to inconsistent scenarios. Thus it is the nature of the concepts themselves that forms the main focus of a universal logic. A further reason such a logic is removed from empirical concerns is its claim to an a priori status, situating it outside any 'web of belief' (Shapiro [2000]). Nonetheless, there is within the universal framework an intersection of our deductive structure with the structure of the way we do in fact reason, and this in turn intersects with the general structure of reality. From the many perspectives we may take of this intersection, naturalism gives one that highlights some features we ought to expect of the deductive structure, or of the nature of deduction itself.

If we adopt such a perspective, then, and so accept for a minute the initial ideal with which the naturalist begins, we note that Maddy's logical naturalism falls some way short of the attainment of this ideal, and indeed of the attainment of any of its humbler versions. This, though, is just in case we take such a picture as applying to classical logic. When we take the same general philosophy and apply it to the logic of Meaning Containment, we move to within grasping reach of the strongest version and to (at least an arguable) attainment of the humbler versions canvassed.

Working with a modernised set of Kantian categories and forms of judgement, Maddy examines how these might 'underlie the laws of logic' on p. 70 of her [2002]. The first such category is that of an object as a member of a class: 'objects grouped together by their relational similarities', also on p. 70 of [2002]. From this, Maddy builds 'minimal' or pre-formal versions of conjunction and disjunction, as 'counterparts of ... intersections and unions'. We pause here to note that the semantics picking out the logic MC, called 'content semantics' in Brady [1996], follows the same conceptual route. The idealisation required to get from these notions to the connectives of classical logic is, as Maddy goes on to show, quite substantial. Meaning Containment, on the other hand, is the natural crystalisation of just such set-theoretic concepts. For example, in Brady [1996], conjunction corresponds to a closed union and disjunction to an intersection of contents.

But as Maddy notes, the problems for classical logic begin with:

> 'extend[ing] the minimal versions [of the fundamental categorical forms] to full [classical] connectives capable of applying to any descriptions' ([2002], p. 71).

The first aspect of this problem that Maddy considers is the contingency of the LEM:

> 'does thinking in terms of [the categories] ... also commit us to granting, for example, that any given object must either have a given

> property or fail to have it? It seems to me [Maddy] that the answer must be no' ([2002], p. 71).

Classical logic's universal LEM is, then, a significant idealization,

> 'required to cover the distance between the rudimentary logic of our fundamental conceptual machinery and the laws of modern logic' ([2002], p. 73).

In fact, Maddy's general naturalistic picture of the growth of logical concepts from pre-formal categories highlights the appropriateness of MC over classical logic in a number of places. The universal LEM is just one of these:

> 'This is the first major idealization . . . Removing all truth-value gaps and restoring the standard truth-functional negation produces the full store of propositional tautologies involving conjunction disjunction and negation . . . a second sort of idealization [is required] . . . in the definition of the conditional . . . there need be no 'causal connection' (Frege [1879], p. 14) between the two components: 'if the sun is shining, then 3 x 7 = 21' is true, though the sun's shining has nothing to do with the arithmetical fact. Here we make a clear and deliberate departure from the content of the underlying category' (Maddy [2002], p. 73).

Put this way, the notion that this is a positive trade off seems, at least, debatable. That is, if we begin with pre-formal notions and look, from a relatively unbiased (non-classical) perspective, at the sort of naturalistic story Maddy presents, then the case for MC seems clear. A logic that preserves our pre-formal conceptualizations *without* the idealizations outlined above, is surely preferable to one necessitating the same. This is particularly true given that none of the essential strengths of classical logic are lost (see footnote 3), nor is the general logical goal — the formalization of deduction — compromised in MC. So, we do in fact have the means of maintaining all of the former, without the substantial idealisation (perhaps properly called a distortion) of a universal LEM and without an unnatural truth-functional negation.

It seems, in short, that Maddy has presented a good argument for the contingency of the LEM, but she goes on to argue that its axiomatic status in classical logic is a necessity. This does not make sense once we know we've an alternative: upon consideration of this alternative, it becomes apparent that the universal LEM is unnatural, both pre-formally and, it turns out, in application. Building a logic by beginning with an unbiased analysis of the pure concepts involved results in a logic closer to the concepts, i.e. one less idealized; one with no obvious 'departure from the content of the underlying category [/concept], motivated by the [perceived, classically biased] needs of logical theorizing' ([2002], p. 73), and one with a more coherent application.

Meaning Containment, with its restricted LEM, embodies none of the radical idealizations required to reach classical logic. The move from the sort of pre-formal categories Maddy proposes to the logic itself is, in the case of MC, an entirely natural one. In fact, Maddy concludes from her analysis that not just the LEM, but the whole of (classical) logic itself is contingent. We reject this conclusion (for the logic MC), but embrace the premises. That is, her naturalistic analysis can justify not only the contingency of some principles, but also the apriority of others.

We predict that the natural fit of the logic MC with the general structure of conceptual thought will be shown by any analysis of pre-formal cognition. Rather than outline possible examples here, we simply note that a naturalistic or Kantian analysis immediately suggests a more inclusive, less idealised logic than classical logic, and this points not only to a restricted LEM, but to Meaning Containment (MC) itself; i.e. to a revised conditional, to content semantics and to a non-classical negation, among other things. One way of characterizing the situation is to note that by such analysis we are led to a clearer picture of just which logical systems deserve the title 'deviant' and which the title 'natural'. Thus, MC is the natural logic, and classical logic shows up as deviant, turning Quine on his head.

Rather, then, than revise the original naturalistic ideal downward to meet the limitations of the proposed (classical) logic, we suggest that the existence of a logic that 'fits' the general naturalistic story both redeems this ideal and lends support to the particular logic (MC) thereby exemplified. Specifically, we note that such a universal logic has a strong claim to Kantian apriority (if indeed anything has): in that it is 'epistemologically independent of sensory empirical experience, and ... a necessary underpinning of empirical science' (Posy [2005], p. 333). The transcendental argument gives the independence of the resultant logic, while its general empirical verifiability, in accordance with an overarching naturalistic philosophy, gives the required agreement with empirical science. There will be further discussion on this in later work.

In the next two sections, we examine the DS and the LEM with a view to working out when we should assume them. Despite not being logical laws, they are both used extensively in everyday and technical reasoning. We need justification for them, over and above the meaning considerations used to justify logical laws.

5. *The Assumption of the DS*

We start by considering the DS in the context of classicality, i.e. in conjunction with the LEM. (See footnote 3.) In [UL], classicality was taken to apply over the physical world and to all abstractions and idealisations thereof. It turns out, though, that this criterion is too broad. Specifically, it applies the notion of classicality where it clearly ought not apply. For example, both Heisenberg's uncertainty regarding the position and momentum of an electron, and the Gödel sentence G of Peano arithmetic fall within its scope. Given that classicality itself is an abstracted notion, effectively limiting the range of 'real' world situations to an idealized subset, the consequent enlargement of this subset to again cover what it was abstracted from in the first place, seems misguided. That is, taking classicality to apply to our broadest concept of 'the physical world' is at best an otherwise unjustified technical convenience, at worst a severe misrepresentation of our knowledge of physical reality.

These are not the only examples of non-classical physical reality. Another could be the status of current information about far-distant space. Attaching a definite truth value to such information seems contrived, as there is no way such information can be checked due to the time taken for light to reach us. By comparison, the admission of such (real) truth value gaps into our formal logic seems natural and more faithful to our knowledge of the concepts concerned. This echoes a similar argument made in Brady and Rush [200x] in which we conclude that classicality can reasonably be taken to apply over (whatever turns out to be) the *large part* of the physical world, and what can be mapped into this. And the derivation of the LEM and the DS is one of the means by which we can expand and refine the concepts concerned: both that of classicality, and that (part) of the physical world.

We now consider the DS by itself. The DS ($\sim$A, A $\vee$ B $\Rightarrow$ B) might be provable using metacompleteness through the Priming Property, together with the simple consistency of A. However, in this case, the DS is not needed as B is independently provable. The DS might also be provable using rejection and the Real DS ($\dashv$ A, A$\vee$B $\Rightarrow$ B), together with the consistency of A in the form $\vdash$ $\sim$A $\Rightarrow\dashv$ A, as set out in Brady [2007]. To make this consistency work, we need rejection-soundness, i.e. for all formulae A, it is not the case that both $\vdash$ A and $\dashv$ A, which is usually easy to show in a rejection system. There is a problem with the broader use of rejection in that no rejection-complete axiomatization for MC (i.e. where, for all formulae A, either $\vdash$ A or $\dashv$ A) has been established. However, rejection-completeness does hold for FDF, the first-degree formula fragment of MC (see Brady [2008]), which is what one is most likely to use in practice, as second degree is rarely used. The proof here relies on normal-forming operations, which are not adequate for the full MC.

When used, the disjunction in the DS must come from somewhere. If proved outright, i.e. without assumption, in a metacomplete system then the DS is not needed under simple consistency, as described above. If the disjunction is derived through the LEM, which is proved without assumption, then each disjunct may well be derivable and the DS not needed, as above. If the disjunction is derived through assumptions, including the LEM, one would have to similarly assume the DS to use it.

Indeed, many uses of the DS are based on a prior use of the LEM or similar disjunctive assumptions. Take, for example, the 9x9 number puzzle, Sudoku, currently occurring in Australian newspapers. It consists of a 9x9 array of squares, broken down into 9 3x3 arrays. In each square is to be put a single number, 1,. . . ,9, such that each 3x3 array has exactly the numbers 1,. . . ,9, and also each row and each column of the 9x9 array has exactly the numbers 1,. . . ,9. All these rules are disjunctive. The fact that, in a particular square (i,j), at least one of 1,. . . ,9 occurs is an extended LEM to nine propositions, which can be expressed thus: $1_{i,j} \vee 2_{i,j} \vee \ldots \vee 9_{i,j}$. The fact that no two distinct numbers can occur, say 1 and 2, can be expressed: $\sim(1_{i,j} \& 2_{i,j})$, which is equivalent to the disjunctive $\sim 1_{i,j} \vee \sim 2_{i,j}$, by De Morgan's Law. For each of the three above kinds of subsets in the 9x9 array, each number occurs at least once. The number 1, say, occurs in either of the squares: (1,1), (1,2), . . . , (1,9), in the first row. This can be expressed disjunctively: $1_{1,1} \vee 1_{1,2} \vee \ldots \vee 1_{1,9}$, using the terminology above. Each number cannot occur in two of the squares in these kinds of subsets. E.g., the number 1 cannot occur in both of the first two of the above squares, expressed: $\sim(1_{1,1} \& 1_{1,2})$, which is equivalent to $\sim 1_{1,1} \vee \sim 1_{1,2}$.

In the Sudoku puzzle, certain squares are already filled in with specific numbers. The object is then to fill in the remaining squares, using the above rules. The puzzle is composed so that there is only one solution. If a number 1, say, is given as being in square (1,2), i.e. $1_{1,2}$, then, by application of the DS to the above disjunction $\sim 1_{1,1} \vee\ 1_{1,2}$, $\sim 1_{1,1}$ follows and the number 1 cannot be in (1,1). This negative statement can then be used to whittle down what started as a nine-fold disjunction, by further application(s) of the DS. All the other cases follow the same logical pattern. As we can see, in making deductions from the given information of the puzzle, the DS is successively applied to these disjunctions until a single number is found for each square, yielding the unique solution, consisting of one 9x9 array of these digits, subject to the rules.

Given the above, there seems to be point in assuming the DS whenever there is an assumption of the LEM. Further, most logicians would assume the DS outright and change their system if a contradiction is found. Though this tactic is usually adopted by classical logicians, it does seem like a reasonable general strategy, even for paraconsistent logicians who may reach an unacceptable conclusion. Take as an example the statement of Gödel's

Theorem: if Peano Arithmetic is simply consistent then it is $\sim$-incomplete. It is taken to be an incompleteness theorem. It is almost never stated in the contrapositive: if Peano Arithmetic is $\sim$-complete then it is simply inconsistent, as an inconsistency theorem.[7] The exception to this general strategy are those paraconsistent logicians who expose inconsistencies, often whilst including the LEM as part of their logic.[8] This would also apply to those relevant logicians who use a strong relevant logic such as R, which contains the LEM, but does not have the DS as a derived rule. However, these logics are at variance with the logic MC, motivated here and elsewhere. What follows is guided by this logic, and its application.

The DS is more readily assumed than the LEM, in that the LEM is more likely to be dubious than the DS. $\sim$-incompleteness can very easily occur, as in the case of future contingent events, for example. Simple consistency is readily assumed for any serious study, and, as above, if a simple inconsistency were to occur one would tend to revise one's assumptions in such a way as to avoid its re-occurrence. Also, as exemplified by the Sudoku puzzle, it seems quite reasonable to apply the DS to disjunctions established through the use of the LEM. So, it seems safe to allow the DS at the very least to follow the assumption of the LEM, giving primary importance to the LEM.

6. *The Reasonable Assumption of the LEM*

So, we proceed to determine when we can reasonably assume the LEM. As in 5 above, it seems reasonable to assume classicality for the large part of the physical world and for whatever can be mapped into it, and this, of course, applies to the LEM too. The finite natural numbers extend what is mappable into the physical world by using the successor operation, primarily because it is unreasonable to stop applying the LEM at any particular finite number. So, we apply it all the way up the infinity of natural numbers. Other examples of inductively specified entities would be treated similarly. So much for the assumption of the LEM; its proof, however, raises a different set of issues.

As with the DS, we can sometimes prove the LEM outright. If the applied system is metacomplete, any proof of the LEM ($A \vee \sim A$) would pass through one of the two disjuncts, A or $\sim$A. This is, of course, fine. Also, within a system with rejection, which is rejection-complete, the rejection form of the LEM, viz. if $\dashv$ A then $\vdash \sim$A, yields the LEM, since if $\vdash$ A then $\vdash A \vee \sim A$

[7] See pp. 46–47 of Priest [2006] for a discussion of this point.

[8] See p. 11 of Priest [2006], in the case of the Heterological Paradox, where the weaker rule, $A \leftrightarrow \sim A \Rightarrow A \& \sim A$, is used instead of the LEM.

and if $\dashv$ A then $\vdash$ A $\vee$ $\sim$A. This too is fine, but the problem of showing rejection-completeness (either $\vdash$ A or $\dashv$ A, for all A) remains.

We could also prove the LEM by Mathematical Induction over the natural numbers. We put Mathematical Induction (MI) in the form of a meta-rule: $(\mathrm{A(m)} \Rightarrow \mathrm{A(m')}) \Rightarrow (\mathrm{A(0)} \Rightarrow \mathrm{A(n)})$, as the entailment $\rightarrow$ is too strong a relationship between properties applying to different numbers.[9] The application of MI in proving the LEM works because it establishes A(n) or $\sim$A(n), and so we move easily to A(n) $\vee$ $\sim$A(n), but it is quite another matter in the more general form:

$$\begin{aligned} (\mathrm{A(m)} \vee \sim\mathrm{A(m)} &\Rightarrow \mathrm{A(m')} \vee \sim\mathrm{A(m')}) \\ \Rightarrow (\mathrm{A(0)} \vee \sim\mathrm{A(0)} &\Rightarrow \mathrm{A(n)} \vee \sim\mathrm{A(n)}). \qquad (*) \end{aligned}$$

This form is problematic in that there could be switching from A(m) holding to $\sim$A(m$'$) holding, or from $\sim$A(m) holding to A(m$'$) holding. This would prevent a nice induction from flowing through, like one from A(m) to A(m$'$), for all m, or from $\sim$A(m) to $\sim$A(m$'$), for all m. However, a course-of-values induction may still work in lieu of $(*)$ in that it allows a proof of A(m) or of $\sim$A(m) from steps earlier than the immediately preceding one, thus allowing A(k) to be used in the proof of A(m), where $\mathrm{k} \leq \mathrm{m} - 2$. But, there is a more fundamental problem with $(*)$ in that the Priming Property might not hold, in which case the disjunction A(m) $\vee$ $\sim$A(m) might hold independently of either of its disjuncts, thus not allowing us to revert to A(m) or to $\sim$A(m) for rule-based deduction.

So, the general assumption of the LEM in the form A(n) $\vee$ $\sim$A(n) with the variable n ranging over the domain of natural numbers ought not always be made, as a proof might not be possible. Take the case where A(0), A(1),. . . ,A(n),. . . all hold but $(\forall\mathrm{n})\mathrm{A(n)}$ is not provable, due to the lack of a recursive procedure for A(n). Then, A(n) $\vee$ $\sim$A(n) would not be provable. So, arithmetic is $\sim$-incomplete, and indeed incompletable due to Gödel's Theorem. The fact that Gödel's Theorem is a classically based theorem does not matter here, as the unprovability of the Gödel sentence G and its negation $\sim$G will still apply, whether the LEM and the DS are assumed in the

[9] Distinct numbers are not relevant to each other. This can be seen from the fact that the number n, say, is abstracted from sets of n numbered items. A distinct number m will be abstracted from sets of m numbered items, which will be disjoint from any set of n items. So, the numbers are abstracted in a quite separate manner. Hence, entailments such as $\mathrm{m} = \mathrm{n} \rightarrow \mathrm{m'} = \mathrm{n'}$ are not justified, and we instead use $\mathrm{m} = \mathrm{n} \Rightarrow \mathrm{m'} = \mathrm{n'}$, which is an instance of Substitution of Identity, a rule which fails to hold as an entailment when extended to logical connectives. The use of the successor operation serves to inductively generate the natural numbers, rather than to establish an entailment relation between them. This is clearly at variance with Meyer's relevant arithmetic (see Meyer [1975]) using the much stronger relevant logic R.

process of proving it or not. Further, axiomatic set theory is extensively ∼-incomplete, in that the standard axiomatisation of Zermelo-Fraenkel set theory leaves open many questions such as Cantor's Continuum Hypothesis. Again, this incompleteness would still apply when the classical logic is recast as the weaker MC. So, we believe the general use of the LEM is a lost cause when based on MC, which in turn seems to indicate that its ubiquitous use elsewhere amounts to an attempt to achieve the unobtainable by plumping up systems to achieve results beyond straight-forward derivability. However, as illustrated above, there is still technical value in *assuming* the LEM to derive negative results such as the unprovability of G and ∼G, which still apply without it.

Indeed, we can set up three levels of LEM involvement. The *first level* of involvement is where the LEM is proved within a system without assuming it, typically from either of its two disjuncts, as happens for metacomplete systems. The *second level* of involvement is the reasonable or rational assumption of the LEM, as we have been arguing for in this section. This would often involve a combination of the first two levels, when proof is combined with rational assumption. The *third level* of involvement is the assumption of the LEM for technical purposes, over and above the second level, as illustrated by the Gödel sentence G above. We call the assumption of the LEM at this level 'syntactic modelling' as it shares with semantic modelling the inclusion of a theory in its model and both sorts are developed for technical reasons. It may turn out that assumptions at the third level can be used to prove results that are more appropriate for the second level, e.g. the use of infinite mathematics may impact on finite mathematics. However, in this event we would hope that the second level assumptions can also be used to prove such results, but this is something for future work.

7. *The Use of the LEM in the Proof of Cantor's Diagonal Argument*

We are now ready to consider Cantor's Diagonal Argument. It is a reductio argument, set in axiomatic set theory with use of the set of natural numbers. We re-construct it in our logic MC, making use of restricted quantification, as set up in Ch. 13 of [RLR2]. We start by assuming a one-one correspondence between the set of natural numbers N and its power set $\mathcal{P}(\mathrm{N})$. Let $\mathrm{f} : \mathrm{N} \to \mathcal{P}(\mathrm{N})$ be such a one-one function. Formally:

$$(\exists \mathrm{f})[(\forall \mathrm{n} \in \mathrm{N})\mathrm{f}(\mathrm{n}) \in \mathcal{P}(\mathrm{N}) \& (\forall \mathrm{S} \in \mathcal{P}(\mathrm{N}))(\exists \mathrm{k} \in \mathrm{N})\mathrm{f}(\mathrm{k}) = \mathrm{S}](**)$$

Since, for the particular f, $\mathrm{f}(\mathrm{m}) \in \mathcal{P}(\mathrm{N})$, then $\mathrm{f}(\mathrm{m}) \subseteq \mathrm{N}$, which in this general context is defined as $(\forall \mathrm{x})(\mathrm{x} \in \mathrm{f}(\mathrm{m}) \to \mathrm{x} \in \mathrm{N})$. If the LEM holds for $\mathrm{x} \in \mathrm{f}(\mathrm{m})$ then, by the derived MC rule: $\mathrm{A} \to \mathrm{B}, {\sim}\mathrm{A} \vee \mathrm{A} \Rightarrow {\sim}\mathrm{A} \vee \mathrm{B}$,

$(\forall x)(\sim x \in f(m) \vee x \in N)$, which is an extensional version of the subset relation. In a context where the LEM fails to hold, we would have to rely on meanings and so the use of '$\rightarrow$' is appropriate. Anyway, by using '$\rightarrow$' in the definition of '$\subseteq$', we can leave it open as to whether an intensional or extension version is appropriate, in accordance with the presence or absence of the LEM.

To represent this function f, we set up the following sample infinite array of 0's and 1's:

	0	1	2	3	4	.	.	.	n	.	.
0	0	1	1	0	0	.	.	.	1	.	.
1	1	0	1	0	1	.	.	.	0	.	.
2	0	1	0	1	1	.	.	.	1	.	.
3	0	0	1	1	0	.	.	.	0	.	.
4	1	0	0	1	1	.	.	.	0	.	.
.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.
m	.	.	.	.	.	.	.	.	1	.	.
.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.

Each row in the above array represents f(m), by putting '0' when $n \notin f(m)$ and '1' when $n \in f(m)$, into the respective column for n. A subset of N can also be obtained by looking down the diagonal, with '0' representing $n \notin f(n)$ and '1' representing $n \in f(n)$, for each n. We then change each '0' to a '1' and each '1' to a '0' down the diagonal, introducing what we will now call the diagonal set D, where '1' represents $n \notin f(n)$ and '0' represents $n \in f(n)$, for each n.

We represent this set by the following two statements:

(1) $\forall n(n \in D \rightarrow n \in N)$, indicating that $D \subseteq N$.

(2) $(\forall n \in N)(n \in D \leftrightarrow n \notin f(n))$, by the above definition of D.

Thus, by (1), $D \in \mathcal{P}(N)$, and hence $(\exists k \in N)f(k) = D$, from $(**)$, by an obvious property of restricted quantification.

For this k in N, by (2), $k \in D \leftrightarrow k \notin f(k)$, and hence $k \in f(k) \leftrightarrow k \notin f(k)$.

We cannot go further than this. To achieve the required contradiction for the purposes of the reductio argument, one needs the LEM in the form, $k \in f(k) \vee k \notin f(k)$. For then, since $k \in f(k) \rightarrow k \notin f(k)$ and $k \notin f(k) \rightarrow k \notin f(k)$, $k \notin f(k)$. Then, since $k \notin f(k) \rightarrow k \in f(k)$, $k \in f(k)$ and hence $k \in f(k) \& \sim k \in f(k)$.

This is along similar lines to the classical argument for Cantor's Theorem, which introduces the above set D, which cannot correspond to any number n because f(n) differs from D, with respect to the number n, i.e. $n \in D \equiv n \notin f(n)$ and hence $D \neq f(n)$. But $D \subseteq N$, $D \in \mathcal{P}(N)$ and there should be some $k \in N$ such that $f(k) = D$, contradicting $D \neq f(n)$. So, for this argument, we not only need the LEM to prove the classical $n \in D \equiv n \notin f(n)$ from (2), but also $A\&{\sim}B \Rightarrow {\sim}(A \rightarrow B)$ to prove $(n \in D\&n \notin f(n)) \Rightarrow D \neq f(n)$ and $(n \notin D\&n \in f(n)) \Rightarrow D \neq f(n)$, and hence $D \neq f(n)$, where $D = f(n)$ is defined intensionally, as for subsets above.[10] The extra rule $A\&{\sim}B \Rightarrow {\sim}(A \rightarrow B)$ is not available in MC, nor should it be. Comparing it with the rejection version of Modus Ponens, $\vdash A, \dashv B \Rightarrow \dashv A \rightarrow B$, the above rule can be seen to depend on the consistency of B, i.e. $\vdash {\sim}B \Rightarrow \dashv B$, and the completeness of $A \rightarrow B$, i.e. $\dashv A \rightarrow B \Rightarrow \vdash {\sim}(A \rightarrow B)$.

Now, getting back to our proof, the crucial question is: does the LEM hold for $k \in f(k)$? Looking at the construction of D, we have not shown that D is recursively generated. Indeed, the function f is an arbitrary one-one correspondence, and this needs to be so because f must be as general as possible to work the reductio argument. That is, if this reductio argument were to work, all possible one-one functions must be included so as to prove that there is no such function, which would then establish that N and $\mathcal{P}(N)$ have distinct cardinalities. And, even if we assume that the function f is recursively generated, it is not at all clear that the LEM can be shown to hold for $k \in f(k)$ by Mathematical Induction, for reasons given in 6. So, this blocks the proof of Cantor's Theorem and leaves open the question of whether there is a one-one function between N and $\mathcal{P}(N)$ or not.

Thus we are replacing a (classically) established result with an open question. This outcome can be understood either positively or negatively. Negatively, it appears to have lessened the overall body of mathematical knowledge. Of course, what we are arguing here is that Cantor's Theorem never actually constituted mathematical knowledge in the first place, so realising this can hardly be a bad thing. Positively, opening a previously 'decided' question creates room for more, perhaps better mathematical knowledge. One such expansion of our knowledge comes via the reiteration of the relative unimportance of completeness. Although this has been noted before, here its relative unimportance is shown up in a specific context: that of stronger guiding principles to 'good' mathematics, namely relevance, entailment and justified use of the LEM and the DS.

So, the incompleteness here is simply one 'side effect' of the central conceptualisation of the logic MC which, we have argued elsewhere, is a faithful formalisation of the crucial elements of deduction and good argumentation. Realising this ought to shift the emphasis in our derivation of mathematical results from completeness (and, in other applications, similarly from consistency) to something more like 'coherence'. The importance of a central, faithful conceptualisation is what has been underscored here: a coherent

[10] Note that if $A \Rightarrow C$ and $B \Rightarrow C$ then $A \vee B \Rightarrow C$ follows in MC, by use of MR1.

theory (in the overarching sense of being held together by a core conceptualisation) is a more accurate guide for knowledge acquisition than a merely complete theory could ever be. Such a realisation itself may be considered a far greater expansion of our mathematical knowledge than Cantor's Theorem appeared to be.

8. *Examination of some Paradoxes*

There is a marked similarity in logical structure between Cantor's Diagonal Argument and some of the familiar paradoxes. We first look at the Paradox of the Barber. As on p. 297 of [UL], we consider a village, in which there is a barber who shaves all and only those who do not shave themselves. In answer to the question 'Who shaves the barber?', we realize that he shaves himself iff he doesn't. Classically, a contradiction can then be derived. As claimed in [UL], the Barber Paradox is a pseudo-paradox because it is just a reductio argument which rejects the existence of such a barber or such a village. This is similar in structure to Cantor's Diagonal Argument in that it is also a reductio argument, in this case assuming the existence of a one-one function and concluding with k being a member of the diagonal set D iff it isn't. So, there is a logical parity between Cantor's Argument and the Barber Paradox.

Richard's Paradox is similar but more subtle. We set this out as on p. 296 of [UL]. There are only denumerably many finite definitions of real numbers, which are then enumerated in some standard manner with the nth one r_n being called the nth Richard number. These are given decimal expansions with $d_{m,n}$ being the mth decimal place of r_n. We are then given a finite definition of a real number r by the expression: 'the real number whose nth decimal place is 1 if the nth decimal place of the nth Richard number is not 1, and whose nth decimal place is 2 if the nth decimal place of the nth Richard number is 1'. Since all the finitely definable real numbers are in the above enumeration, r is indeed r_k, for some k. Then, $d_{k,n} = 1$ iff $d_{n,n} \neq 1$, and $d_{k,n} = 2$ iff $d_{n,n} = 1$, for all n. For the natural number k, $d_{k,k} = 1$ iff $d_{k,k} \neq 1$, and $d_{k,k} = 2$ iff $d_{k,k} = 1$. Given that 1 and 2 are the only numbers involved, we can put this in the form: $d_{k,k} = 1$ iff $d_{k,k} \neq 1$. This then takes the same logical shape, i.e. $A \leftrightarrow \sim A$, that occurs at the end of Cantor's Argument. As pointed out in [UL], this paradox assumes that every finite expression that purports to define a real number does in fact define one. Indeed, the quoted expression, although finite in length, does involve the (infinite) set of all Richard numbers in its definition, and may well not define a real number.

We next turn to Cantor's Paradox, i.e. the set V of all sets has a cardinality strictly less than that of its power set $\mathcal{P}(V)$, and $\mathcal{P}(V)$ has a cardinality less than or equal to that of V. Without Cantor's Theorem applied to V, this proof does not go through. The cardinality of the set V would be less than or equal to that of $\mathcal{P}(V)$, due to a mapping of V into $\mathcal{P}(V)$, leaving the possibility that V and $\mathcal{P}(V)$ have the same cardinality. The proof of this would use the Schröder-Bernstein Theorem, which would need checking to see if it applies, especially at V.

Given that Cantor's Theorem is left as an open question, Skolem's Paradox is also worth examining. The Skolem-Lowenheim Theorem states that every first-order theory has a denumerable model. But, by Cantor's Theorem, set theory has non-denumerable sets, which thus have only denumerably many elements in a denumerable model. This seeming contradiction is Skolem's Paradox, which is resolved by distinguishing object-language-provable non-denumerability of a set from meta-theoretically determined denumerability of the number of elements in the domain of a model. Nevertheless, the Paradox could be reconciled by a set theory with at most denumerably infinite sets, obtained by adding the negation of Cantor's Theorem, i.e. that all sets can be put in a one-one correspondence with a subset of the natural numbers. Indeed, it would be interesting if a denumerable model could be found for such a set theory with its internal cardinalities matching up with its meta-theoretically determined cardinalities, thus ensuring the consistency of the addition of the negation of Cantor's Theorem and the independence of Cantor's Theorem. The size of these corresponding meta-theoretically determined sets would be found by counting the members of the domain that correspond to members of the object-language set. We need to do further work here, and this will likely be undertaken in a subsequent paper.

9. *Examination of a 'Negation-free' Proof of Cantor's Theorem*

Although the proof of Cantor's Theorem, given in 7, is the standard one, there are sometimes claims of a positive proof or at least one that does not use the LEM. One such proof of Cantor's Theorem — a very clever one based on Yablo's Paradox — is due to Raja in [2005], which claims to be negation-free.

Raja assumes the one-one function M: $X \rightarrow \mathcal{P}(X)$, where X is an arbitrary set and $\mathcal{P}(X)$ is the power set of X. He defines a trace $\{s_0, s_1, s_2, \ldots\}$ such that $s_0 \in X$ and, for $j > 0$, $s_j \in M(s_{j-1})$. He defines $t \in X$ to be a simple element, if all possible traces beginning with t terminate, and termination occurs at s_j iff $M(s_j)$ is empty, for j finite. He then defines $N = \{t \in X : t$ is a simple element$\}$.

Raja then claims that N, a subset of X, cannot lie in the range of M. For, if there is an $n \in X$ such that $M(n) = N$, then n is simple as all its traces terminate. Thus, $n \in N$ and n is no longer a simple element as the trace {n, n, n, ... } does not terminate, contradicting n being a simple element. Hence, the set is outside the range of N, as claimed.

The issue here is whether N is empty or not, that is, whether there are any simple elements or not. This is an instance of the LEM and, as above, since M is an arbitrary one-one function, M need not be recursive and this issue need not be recursively determined. So, the LEM need not hold. Now, if N has an element then put $M(n) = N$, for some $n \in X$, and, by Raja's argument, n is simple. As argued further, $n \in N$ and n is not simple, a contradiction. Alternatively, if N is empty then putting $M(n) = N$, for some $n \in X$, n is simple with a one-element terminating trace $\{n\}$, which is again a contradiction, but a different one. So, in deriving both these contradictions, we are using the above LEM. Raja does not consider the null case since, in the conduct of his argument, he assumes there are simple elements to construct his trace. Thus, his proof is not really negation-free and he needs to assume the LEM in the process.

10. *Mathematical Development With a Restricted Assumption of the LEM*

Recall, as stated in 7, that it is not clear that the LEM can be shown to hold for $k \in f(k)$, even if f is recursive. We are faced, then, with a new question: specifically, the question as to where we can draw a suitable line in the key mathematical theories of arithmetic and naive set theory, where the LEM ought to hold on one side and may well fail on the other. Along with our intuitions, we have the following three theorems, which will greatly help in the process of finding this question an answer.

Theorem 1.
If the LEM holds for all the atomic expressions of a theory based on MC then the LEM holds for all compound formulae of the theory, built up from the atoms by using only $\sim$, & and $\vee$.

Proof. We inductively consider each connective in turn.

$\sim$. If $A \vee \sim A$ holds then so does $\sim A \vee \sim\sim A$, by $A \rightarrow \sim\sim A$ (derivable in MC) and A1,5–7.

&. If $A \vee \sim A$ and $B \vee \sim B$ hold then so do $(A \vee \sim A \vee \sim B)\&(B \vee \sim A \vee \sim B)$, $(A\&B) \vee \sim A \vee \sim B$ and $(A\&B) \vee \sim(A\&B)$. Use is made of A5, commutativity and associativity of '$\vee$', R2, distribution of '$\vee$' over '&' and De Morgan's Law, all of which are available in MC.

$\vee$. If $A \vee \sim A$ and $B \vee \sim B$ hold then so do $(A \vee B \vee \sim A)\&(A \vee B \vee \sim B)$, $A \vee B \vee (\sim A\&\sim B)$ and $(A \vee B) \vee \sim(A \vee B)$. This follows similarly to the &-case. □

Theorem 2.
If the disjunctive form of the DS $(C \vee \sim A, C \vee A \vee B \Rightarrow C \vee B)$ holds for all the atomic expressions of a theory based on MC then this form of the DS holds for all compound formulae of the theory, built up from the atoms by using only $\sim$, & and $\vee$.
[Note that the disjunctive form of the DS is interderivable with the ordinary DS. For one way we use MR1, and for the other we substitute B for C.]
Proof. As for Theorem 1, proof is by induction on formulae.
$\sim$. Let $D \vee \sim A, D \vee A \vee B \Rightarrow D \vee B$ hold, for any D. Assume $C \vee \sim\sim A$ and $C \vee \sim A \vee B$. By A9, A5 and commutativity and associativity of '$\vee$', $(C \vee B) \vee \sim A$ and $(C \vee B) \vee A \vee B$. By putting $C \vee B$ for D, $(C \vee B) \vee B$ and hence $C \vee B$.
&. Let $G \vee \sim A$, $G \vee A \vee C \Rightarrow G \vee C$ hold, for any G, and $H \vee \sim B$, $H \vee B \vee C \Rightarrow H \vee C$ hold, for any H. Assume $E \vee \sim(A\&B)$ and $E \vee (A\&B) \vee C$. Then, by a De Morgan's Law and distribution, $E \vee \sim A \vee \sim B$, $E \vee A \vee C$ and $E \vee B \vee C$. Hence, $(E \vee \sim B) \vee \sim A$ and $(E \vee \sim B) \vee A \vee C$, and, putting $E \vee \sim B$ for G, $E \vee \sim B \vee C$ follows. So, $(E \vee C) \vee \sim B$ and $(E \vee C) \vee B \vee C$, and, putting $E \vee C$ for H, $(E \vee C) \vee C$ and hence $E \vee C$.
$\vee$. Again, let $G \vee \sim A$, $G \vee A \vee C \Rightarrow G \vee C$ hold, for any G, and $H \vee \sim B$, $H \vee B \vee C \Rightarrow H \vee C$ hold, for any H. We assume $E \vee \sim(A \vee B)$ and $E \vee (A \vee B) \vee C$. Then, as above, $E \vee (\sim A\&\sim B)$, and hence $E \vee \sim A$ and $E \vee \sim B$. So, $(E \vee B) \vee \sim A$ and $(E \vee B) \vee A \vee C$, and, putting $E \vee B$ for G, $(E \vee B) \vee C$ follows. So, $E \vee \sim B$ and $E \vee B \vee C$, and, putting E for H, $E \vee C$ follows. □

We also prove the following important result which shows that Gödel's Theorem cannot be proved for formulae without quantifiers. We will need the following Peano Axioms, expressed in a suitable form for our logic:
1. $m = n \Rightarrow m' = n'$.
2. $\sim 0 = n'$.
3. $m' = n' \Rightarrow m = n$.
4. $\sim m = n \Rightarrow \sim m' = n'$.

We assume the basic properties of '=', such as reflexivity, symmetry and transitivity, prior to setting up these axioms.

As explained in footnote 9, we put Axioms 1 and 3 in rule form because there is no common content between a number and its successor to establish the entailment between them. Also, we need to add Axiom 4, as the converse rule-form of Axiom 3, which, though derivable classically and in Meyer's relevant arithmetic, is not likely to be derivable here. In a logic such as MC, which is based on the 4-valued logic E_{fde} of ENT1, and without the LEM and DS, we need to specifically add negative as well as positive properties to fill out our axiomatized concepts to maintain the balance between the positive and the negative.

Theorem 3.
For all formulae A, with no quantifiers and no occurrences of '$\rightarrow$', if $A \vee {\sim}A$ is provable in Peano Arithmetic, based on the logic MC and on the Axioms 1–4 above, then, for every constant instance A' of A, either A' or ${\sim}A'$ is also provable.
Proof. If $A \vee {\sim}A$ is a theorem, so is $A \vee {\sim}A \vee B$, for any B. We use this property to set up the following formula induction on such formulae A that are built up from atoms by using $\sim$, & and $\vee$ only. For any B, for these formulae A, we prove that if $A \vee {\sim}A \vee B$ is a theorem in Peano Arithmetic using the above Axioms 1–4, based on the logic MC, then, for every constant instance A' of A, either A' or ${\sim}A'$ is also provable. By a constant instance, we mean a formula obtained by replacing all free variables over natural numbers by specific natural numbers, i.e. by 0 or by a particular successor of 0.
(i) For the base case, we can show that a=a, for any specific natural number a, by reflexivity of '='. If the specific natural numbers a and b are distinct with a non-zero difference of c, we obtain ${\sim}0 = c$ by Axiom 2, ${\sim}c = 0$ by symmetry of '=' if needed, and, by applying Axiom 4 successively, we derive ${\sim}a = b$.
(ii) We assume the property for A and prove it for ${\sim}A$. So, for any B, if $A \vee {\sim}A \vee B$ is a theorem, then, for every constant instance A' of A, either A' or ${\sim}A'$ is also a theorem. Let ${\sim}A \vee {\sim}{\sim}A \vee C$ be a theorem and let ${\sim}A'$ be a constant instance of ${\sim}A$. Then, $A \vee {\sim}A \vee C$ is a theorem and A' is a constant instance of A. By the induction hypothesis, either A' or ${\sim}A'$ is a theorem. So, either ${\sim}A'$ or ${\sim}{\sim}A'$ is also a theorem.
(iii) We assume the property for both A and B, and prove it for A&B. So, if $A \vee {\sim}A \vee C$ is a theorem, then, for every constant instance A' of A, either A' or ${\sim}A'$ is a theorem. Also, if $B \vee {\sim}B \vee D$ is a theorem, then, for every constant instance B' of B, either B' or ${\sim}B'$ is a theorem. Let $(A\&B) \vee {\sim}(A\&B) \vee E$ be a theorem and let $(A\&B)'$ be a constant instance of A&B. Then $(A\&B)'$ can be put as $A'\&B'$, where A' is a constant instance of A and B' is a constant instance of B. By De Morgan and distribution properties of MC, $A \vee {\sim}A \vee ({\sim}B \vee E)$ and $B \vee {\sim}B \vee ({\sim}A \vee E)$ are also theorems. By the induction hypothesis, either A' or ${\sim}A'$ is a theorem and either B' or ${\sim}B'$ is a theorem. Hence, either $A'\&B'$ or ${\sim}A'$ or ${\sim}B'$ is a theorem, in which case either $A'\&B'$ or ${\sim}(A'\&B')$ is a theorem, through use of disjunction introduction and De Morgan's Law.
(iv) We similarly assume the property for both A and B, and prove it for $A \vee B$. We let $(A \vee B) \vee {\sim}(A \vee B) \vee E$ be a theorem and let $(A \vee B)'$ be a constant instance of $A \vee B$. By MC, $A \vee {\sim}A \vee (B \vee E)$ and $B \vee {\sim}B \vee (A \vee E)$ are theorems. By induction similar to that in (iii), A' or B' or ${\sim}A'\&{\sim}B'$ is a

theorem, whereupon either $A' \vee B'$ or $\sim(A' \vee B')$ is a theorem.
As at the beginning, if $A \vee \sim A$ is a theorem, so is $A \vee \sim A \vee B$, for any B. Then, by the property just proved, for every constant instance A' of A, either A' or $\sim A'$ is also a theorem. □

Getting back to determining the scope of application of the LEM, we briefly examine Peano arithmetic and naive set theory, as both have some imput into Cantor's Argument. We first consider Peano arithmetic. As we have argued for in 6, the LEM can reasonably apply to constant arithmetic statements and to those involving variables, which have been recursively established. However, we can gain some further insight from Theorems 1, 2 and 3. From Theorem 1, we can see that if the LEM is assumed for atomic formulae of the form $m = n \vee \sim m = n$, then the LEM can be proved for all arithmetic statements built up from these atoms by using $\sim$, &, and $\vee$. Theorem 2 shows a similar result for the DS, so the two can work in tandem. So, if classicality is assumed for the atomic expressions, classicality can be proved for all formulae built using only '$\sim$', '&' and '$\vee$'. However, it can be shown that, if the logic MC was extended to MCQ, which is QMC plus the two distribution properties, $\forall x(A \vee B) \rightarrow A \vee \forall xB$ and $A\&\exists xB \rightarrow \exists x(A\&B)$, this result will extend to all quantificational formulae without '$\rightarrow$'. Such formulae would then include the Gödel sentence G and its negation, neither of which are provable, whether the logic is based on MC or is classical. Nevertheless, there is a good reason to reject these two distribution properties and this undesirable consequence, which we pursue below.

Theorem 3 shows that if the LEM, $A \vee \sim A$, is provable in Peano arithmetic for these formulae, without quantifiers or '$\rightarrow$', then, for every constant instance A' of A, either A' or $\sim A'$ is also provable. This enables us to assume the LEM for the atomic sentences m=n with impunity, knowing that the problem faced by Gödel's Theorem cannot occur for these formulae. Indeed, Theorem 3 proves a very desirable property for us, ensuring that whenever the LEM holds for these formulae, it is based on instantiations of either disjunct. Further, if A is a constant formula (i.e. a sentence), either A or $\sim$A will be provable. So, we assume henceforth that the LEM applies to formulae, without quantifiers or '$\rightarrow$'. We also assume the DS for these formulae, as argued for in 5 and through the application of Theorem 2. Beyond this, the use of the LEM will be proved through one of its disjuncts or by induction. The DS can be further assumed as required, as argued in 5.

We could ensure extensionality by using restricted quantification, restricting it to a finite set of natural numbers. However, we would need to introduce some set theory in order to do this, and then there is the problem, mentioned previously, about deciding which number is the maximum for such a purpose. We could also stick with predicates that hold for only finitely many numbers, but this is too restrictive as most that one would want to deal with would hold for infinitely many numbers. So, it is better to proceed with infinite quantification over all natural numbers.

Moreover, quantification over an infinite set can harbour non-recursive properties which, by their nature, are not provable. This situation creates a $\sim$-incompleteness, as obviously the negation of a non-recursive property is not going to be provable either. Further, this puts a limit on the desirable use of the LEM and hence of extensionality. So, such quantifiers should be supported by the intensional meaning containment, rather than through extensional means, involving conjunction, disjunction, the LEM and the DS.

This brings us to the distribution properties, $\forall x(A \vee B) \rightarrow .A \vee \forall xB$ and $A\&\exists xB \rightarrow \exists x(A\&B)$. In all the first author's earlier work, both of these properties were included as axioms in the quantificational extensions LQ of all of my sentential relevant logics L (see Brady [1984a], in particular). However, they were initially introduced by Anderson in [1960], for his quantified entailment logic EQ, where the second property followed from the first using the definition: $\exists xA =_{df} \sim\forall x\sim A$. However, since the sentential distribution properties, $(A \vee B)\&(A \vee C) \rightarrow A \vee (B\&C)$ and $A\&(B \vee C) \rightarrow (A\&B) \vee (A\&C)$, both hold, the above distribution properties hold when the quantification is over a finite domain.

In the first property, $\forall x(A \vee B) \rightarrow .A \vee \forall xB$, there is a use of disjunction in supporting the second universally quantified statement $\forall xB$, and so it is a case of extensional support for a universal statement which needed intensional support, as we argued above. This extensional support can be seen more clearly from its deductive equivalent, $\forall x(A \vee B) \rightarrow .\exists xA \vee \forall xB$, where x can be free in A. This can in turn be transformed into $\forall x(A \supset B) \rightarrow .\forall xA \supset \forall xB$, where $A \supset B =_{df} \sim A \vee B$. Not only is it odd to have an entailment between two '$\supset$' statements, but we can also see the '$\supset$' support for the $\forall xB$. Indeed, Mares and Goldblatt in [2006] appropriately refer to the first property as 'extensional confinement', the second universal quantifier being confined by the disjunction. The second property, $A\&\exists xB \rightarrow \exists x(A\&B)$, can be established from the first by contraposition and $\exists xA \leftrightarrow \sim\forall x\sim A$. And, there is also a usage of extensional conjunction in relating the existentially quantified B to that of A&B. Note that all the other quantificational distribution properties are establishable primarily using the intensional '$\rightarrow$', together with sentential properties.

Further support for dropping $\forall x(A \vee B) \rightarrow .A \vee \forall xB$ can be obtained from Dummett in [1977], p. 31, and Beall and Restall in [2006], p. 64–65, both of whom argue from an intuitionist standpoint. Quoting Dummett:

> 'Then to have a proof of $\forall x(Fx \vee A)$ is to have an effective operation which can recognize as associating to each number n a proof either of Fn or of A. However, since there are infinitely many cases

> to consider, we cannot in general tell whether the operation will ever actually provide a proof of A, or will provide a proof of Fn for every n; we are therefore not, in general, in a position to assert either A or $\forall$xFx, and have no guarantee that we shall be in such a position after any finite number of applications of the operation which constituted the proof of $\forall x(Fx \vee A)$.'[11] [We have not employed Dummett's distinction between numerals and numbers here.]

This argument would apply for us as well.

Using the form $\forall x(A \vee B) \rightarrow .\exists xA \vee \forall xB$, Beall and Restall on p. 65 use the example where all students attending a logic subject have either done the prerequisite or have got special permission to attend. They argue that $\exists xA \vee \forall xB$ is not generally justified at any stage of asking the students one by one. We can extend this type of example to an infinite set for our purposes, where a universal disjunctive property holds by assumption or deduction. However, there may be no rhyme or reason for the first or second disjunct of the property to hold in individual cases.

So, we use the logic QMC, without the two distribution properties, rather than MCQ, following the terminology in Mares and Goldblatt [2006] for QR. Indeed, their semantics for the logic QR, given in their [2006], is not only more straightforward than that for RQ but also has semantic features that make due sense from our perspective. Their semantics for QR builds on the Routley-Meyer semantics for R, with its extensional truth-functional features, but adds propositions to the semantics so that the truth of a universally quantified statement at a world is obtained by entailment from a proposition which is true at the same world. So, a universal statement is intensionally supported through an entailment, rather than being obtained through an extensional truth-functional extension of the sentential logic using domains, as happens in Fine's complex semantics for quantified relevant logics in his [1988].

We next move on to examine naive set theory. Can we draw a similar line for the LEM here? Again, as in 6, the LEM can reasonably apply to constant finite set-theoretic statements, and this is the second level of LEM involvement. It can be extended to those statements involving variables, which have been recursively established over the infinite set of natural numbers. Here, we would need the addition of the Axiom of Infinity, $\exists x(\varnothing \in x \,\&\, (\forall y \in x) y \cup \{y\} \in x)$, introducing the set ω of canonical natural numbers. This may well extend to those statements, involving variables, which have been established by transfinite induction, with the addition of the

[11] I owe this reference to Lloyd Humberstone.

cumulative hierarchy of canonical ordinals. This is likely to include statements of the form $x \in a$, where a is a specific member of the cumulative hierarchy.[12]

This is probably about as far as one would go with the LEM, without encroaching on the general atom $m \in n$. Note here that the general assumption of $m \in n \vee {\sim}m \in n$ will lead to the LEM holding for the Russell set R, by simple substitution, and a contradiction, $R \in R \,\&\, {\sim}R \in R$, in the form of Russell's Paradox will then be derivable from $R \in R \leftrightarrow {\sim}R \in R$ through $(R \in R \vee {\sim}R \in R) \rightarrow (R \in R \,\&\, {\sim}R \in R)$. As discussed in 5, such a simple inconsistency should not occur, at least in areas covered by the LEM, and nevertheless should not occur in serious study.

Also, for any instances of $m \in n$ for which the LEM holds, Theorem 1 ensures that the LEM holds for all formulae compounded from them using '$\sim$', '&' and '$\vee$'. Theorem 2 does the same thing for the DS. Theorem 3 would be a good meta-theorem to have for applied systems in general, including set theory, but we would have to restrict it here to apply to those atomic instances, $m \in n \vee {\sim}m \in n$, of the LEM which hold and, of course, such that, for each of their constant instances, one of the disjuncts is provable.

This is an on-going project and these ideas for the above two theories will be pushed further and into other areas of mathematics, as the project proceeds.

11. *Comparison with Intuitionist Mathematics*

We finish by briefly comparing our treatment using MCQ with intuitionist and recursive mathematics. Though intuitionist logic and MC are both constructive logics, their philosophical backgrounds and thus their logics are very different. For intuitionists, mathematical truths are intuited and synthetic *a priori*, as well as mentally constructed. This in turn induces a constructive logic. Indeed, intuitionists permit only potential infinities. They construct their mathematics first and then apply the principles therein to logic. This process is all set out in Heyting's book [1966], which finishes with an axiomatization of intuitionist logic. It is not a relevant logic in that $A \rightarrow .B \rightarrow A$ and ${\sim}A \rightarrow .A \rightarrow B$ are axioms and negation can be asserted just when a derivation of a contradiction from the unnegated statement is made. It follows that only about half of the double negation, contraposition and De Morgan principles hold, and that the LEM is not a logical truth. On

[12] In [UL], using classical meta-logic, I have used transfinite induction to show that naive set theory is simply consistent. Such usage is appropriate as it is an unprovability result that would still hold, with or without the assumption of the LEM and the DS in the meta-theory. As it is an addition for technical purposes, this would be at level three of LEM involvement.

the other hand, its positive sentential fragment is the same as that of classical logic.

In comparison, QMC is a weak relevant logic with a De Morgan negation, meaning that all of the double negation, contraposition and De Morgan principles hold. The LEM is also not a logical law. Its positive sentential fragment is properly contained in that of classical and intuitionist logic. One sense in which it is constructive is that it recognizes logic as being about inferences from premises to conclusion, with its inferences being driven by meaning containment rather than by truth. The infinite can be introduced, but its properties are generally determined by meaning rather than by extensionality. However, some inferences do remain which would normally be judged as being non-constructive. For example, the De Morgan analogue, $\sim\forall x\sim A \rightarrow \exists xA$, holds in QMC but fails in intuitionist logic.

According to Posy, Brouwer's analysis of the problems with classical logic and classical mathematics involved the proposal that the LEM originated with an early mathematics dealing only with finite domains. For these early reckonings, Brouwer said, the LEM was in fact valid: 'mathematical objects were effectively complete' (Posy [2005], p. 334). But with infinite domains came incompleteness and the invalidity of the LEM. Mathematicians, though, 'continued to apply the old logic as if nothing had changed' (Posy [2005], p. 334), which is why, at least according to Brouwer, classical logicians and mathematicians still persist in their error of 'assert[ing] the existence of things that don't in fact exist' (also in Posy [2005], p. 334). We reason similarly; except that in our case, infinite domains may rightly be asserted to exist, but these domains may not be extensional, and so such an assertion may need an intensional justification.

Interestingly, as Posy points out, the ontological emphasis here can also inform a more general rejection of the LEM, even across ordinary empirical domains. This is due to the entirely reasonable notion (already noted above) that, even in ordinary empirical situations, there is much we can not know, and may conceivably never, decide. Of course, given our concern stated earlier with what is humanly possible, we focus on the latter situations, taking these to be clear cases of LEM failure, given that, in these cases, decidability is ruled out altogether — falling literally beyond our conceptual scope.

Under an intuitionistic interpretation, there is a (both actual and potential) 'backward' spread of the LEM prohibition from such cases to less obvious cases or cases in which it seems the LEM should hold, and this is one of the problems we hope to have avoided here. We take the LEM to be a reasonable assumption over certain domains, and to hold wherever it can be proved. That is, there seems quite clearly to be a domain in which the LEM may reasonably be taken to hold. This, loosely put, is the large part of the physical world of middle sized objects, to which, indeed, most 'classical' principles still apply in an everyday/common sense fashion — i.e mostly without obvious problems. It is as we examine the far edges of this domain: the very small or the very large or even the very vague, that we have cause to examine and restrict our usage of otherwise workable principles. Indeed, the three levels of LEM involvement, given in section 6, offer an (at least relatively) non-ad-hoc rationale for the application or otherwise of the LEM, guided both by considerations internal to the logic MC, and by the broader concepts MC itself fleshes out and exemplifies.

There is, in comparison, some internal dissonance in the intuitionist stance. There, the LEM can be shown to hold for a property or relation relative to a certain set or domain. This property or relation is then *decidable*. Some 'backward' justification is required here, some rationale for drawing a line between where the LEM prohibition holds and where it may be lifted. But there is, it seems, no obvious or natural means to provide this within the intuitionist framework. There is a comparison to be made with classical logic here: classical logic's universal use of the LEM can be compared with the intuitionist's universal rejection of the same — both are radical idealizations which run into problems at their 'edges', in that they both have difficulty accounting for the exceptions to these idealizations in a non-ad hoc way. Of course, intuitionism is guided by mathematics, and this does provide a framework in which the limits of the application of the LEM can be established. By contrast, from the outset, MC allows for scenarios both wherein the LEM holds, and wherein it does not. The boundaries between these cases may shift upon consideration of individual cases (this is another feature MC shares with intuitionism, namely the ability to assess individual cases as they arise according to an overarching framework), but the three levels of LEM involvement associated with MC ensure that its applicability or otherwise is a neither ad hoc nor over-idealised phenomenon.

12. *Other Problems with Cantor's Argument*

Intuitionism in fact gives us two problems with Cantor's argument. The first we've already touched on — i.e. it oversteps the reach of the LEM. Not much more needs to be said about this, except that this has to do with the relationship of the LEM to existence: the LEM does not apply where there is no construction (see Posy [2005], p. 334), and with the notion that we cannot propose the existence of a new cardinal number bigger than $\aleph_0$, rather 'the species thus constructed are [according to Brouwer] denumerably unfinished' (Posy [2005], p. 323).

The second problem with Cantor's argument, from the perspective provided by intuitionism, is that it compromises an important understanding of the continuum, specifically, the notion of the continuum as viscous. It is a straightforward consequence of Brouwer's continuity theorem that the continuum cannot be split (Posy [2005], pp. 345–6). Posy points out that Cantor had to abandon any conception of the continuum as viscous in order to compose it from a collection of points. Now, while acceptance of Cantor's argument does not entail the rejection of a viscous continuum, it seems that the acceptance of the continuum as viscous does encourage a re-think of Cantor's argument. This is not only because Brouwer already seems to have thought along these lines, but also because the conclusion of Cantor's argument seems less compelling when set against a presumption of viscosity. When we consider the set generated by the diagonalization against the presumption of viscosity, we have an intuitive means of conceptualizing that this new set does not enlarge the original, or take us somehow beyond what we can count. This is because now we can imagine that the new set simply shows that Cantor's particular carving up has residual elements 'clinging to the knife' (Posy ([2005], p. 347). Given this imagery, the proper conclusion becomes that an artificial, imposed structure cannot give a full account of a viscose domain.[13]

The notion of viscosity, then, is neatly formally expressible as follows:
Theorem: Every decidable subset of R either is empty or includes all of R. (R is indecomposable.) (McCarty [2005], p. 368.)
This is proved in much the same way as is the invalidity of the universally generalised LEM (a full discussion and formal treatment of the connection between these two theorems and their proofs is given in McCarty [2005]).

Heyting (who gives the same results, found in his [1966], p. 40) concluded from this that the continuum is not denumerably infinite. Indeed, our own attempted argument towards Cantor's Theorem does go through intuitionistically. And, this argument can be taken further to conclude Cantor's Theorem itself, due to the presence of both $(\mathrm{A} \rightarrow \sim\mathrm{A}) \rightarrow \sim\mathrm{A}$ and $(\mathrm{A} \rightarrow \mathrm{B}\&\sim\mathrm{B}) \rightarrow \sim\mathrm{A}$ in intuitionist logic.[14]

Brouwer's conclusion, by comparison, focuses on existence. From his continuity theorem, and from his acceptance of Cantor's diagonal argument (as a proof that the real interval is uncountable), he concludes that the *existence* of an uncountable cardinal number cannot be shown (Posy [2005], p. 334). Brouwer's perspective, as Posy points out, is highlighted by his division between the study of 'splittable' or separable sets and the study of the continuum. The former is even given a special name: 'separable mathematics' (Posy [2005], p. 346). Thus, Brouwer's interpretation, in particular,

[13] Note that Brouwer showed how to construct the continuum from choice sequences as autonomously existing entities, thus showing that 'we can have a viscous continuum and make it out of points as well' (Posy [2005], p. 347).

[14] I owe this point to Lloyd Humberstone and Bob Meyer.

offers a perspective on the nature of the continuum, as *it exists*, or, to state the same point without metaphysical bias, *if* it exists.

A further perspective on the problems with Cantor's argument takes the argument's impredicativity as its starting point:

> A complaint . . . is that the proof is problematic not for want of explicitness but because it is impredicative. This is because the diagonalizing function f maps down the hierarchy, so that f(s) is a set of lower birthday than s. . . . This is the reason why the proof of uncountability is unacceptable to the constructivist: the objection is that although we have defined f(s) explicitly, we have only done so in terms of s. In truth, the point might be better put by saying that the definition of the real numbers is *impredicative, since the proof of uncountability does no more than exploit the definition*. (Potter [2004], pp. 137–8, our emphasis). [Here, s is the hierarchy of real numbers, ordered by some 1–1 function and f picks out the altered diagonal, creating a real number.]

This accords with our account, although we are not concerned with impredicativity itself, in that we accommodate circularity, provided it is kept intensional.

Indeed, the observation that Cantor's argument is impredicative, standing alone, does not offer much insight, since impredicative definition is a largely accepted mathematical practice. But the same observation bears fruit when accompanied by an analysis of predication itself. Such an analysis is provided throughout Feferman [1998], which includes work on the reach of predicative mathematics. But for our purposes it is enough to note some of the core objections arising from predicativity as a general stance, to the issues raised here.

These include Poincare's 'diagnoses' (in Feferman [2005], p. 591) of typical paradoxes, including the Richard paradox involving Cantor's diagonalization:

> According to Poincare, in this case [Richard Paradox] the vicious circle lies in trying to produce the object r in D by reference to the supposed totality of objects in D; indirectly, then, r is defined in terms of itself, as one of the objects in D. Poincare's second diagnosis is . . . that the source of each paradox lies in the assumption of the 'actual' or 'completed' infinite (in Feferman [2005], p. 591).

Again, though, note that we are not concerned by circularity or by vicious circles since we use the logic to avoid problems with these, but we are concerned by the reach of extensionality. Our own position on Richard's paradox is in accordance with these concerns, as has been stated above (section 8).

Russell, though, took Poincare's diagnosis seriously and as a result disallowed the LEM under universal quantification, but allowed it in variable form, taking the latter to (legitimately) make a claim about 'any' and the former form as an illegitimate claim about 'all' (Feferman [2005], p. 593). Feferman discusses Russell's stance in some detail:

> The importance of this distinction [between 'all' and 'any'] for Russell has to do with the injunction against illegitimate totalities. In particular, with p a variable for propositions, he would admit ... $p \vee \sim p$, but not the statement $(\forall p)(p \vee \sim p)$ ... similarly for properties $P(x)$; significantly, Russell pointed out that the proposed definition of the natural numbers in the form "'n is a finite integer" means "whatever property $\varnothing$ may be, n has the property $\varnothing$ provided $\varnothing$ is possessed by 0 and by the successors of possessors"' ... (van Heijenoort [1967], p. 159). That is, in symbols,
>
> $$F(n) =_{df} (\forall\varnothing)[\varnothing(0)\&(\forall x)(\varnothing(x) \supset \varnothing(x')) \supset \varnothing(n)](***)$$
>
> cannot be replaced by dropping the universal quantifier over properties '$(\forall\varnothing)$'.
>
> ... [Russell] faced the question of when it is legitimate to apply universal quantification over any kind of object, and here he veered away from Poincare's injunction against the 'actual' infinite:
>
> [the reason we can] speak of 'all men' ... is not finitude, but what may be called logical homogeneity. This property is to belong to any collection whose terms are all contained within the range of significance of some one function. It would not always be obvious at a glance whether a collection possessed this property or not (van Heijenoort [1967], p. 163).

By comparison, recall that, in our case, for A without quantifiers, (1) $(\forall x)(A \vee \sim A)$ is acceptable, but (2) $(\forall x)A \vee \sim(\forall x)A$ is not. Similarly, as stated earlier, the general assumption of the LEM in the form (3) $A(n) \vee \sim A(n)$ with the variable n ranging over the domain of natural numbers is not automatically, or generally acceptable. This is comparable with Russell's observation above that ‘it is not always obvious at a glance’ to determine whether an infinite collection possesses a given property. In particular, we argue that infinite and non-recursive properties over an infinite domain require an intenstional justification. That is, only where one or the other conjuncts of the LEM can be shown to hold universally for a particular property n, can the versions (2) and (3) hold, and this is something that needs to be shown case by case (for each n). But this process cannot generally be reversed. From a universal LEM (1), we cannot always speak universally of one of its disjuncts.

Thus, for us, talk of ‘all men’ (and, more pertinently, of all natural numbers) is acceptable, as is the application of the LEM to such collections. It is talk of the LEM holding for ‘all properties of all men/numbers’ that is problematic. That is, it is when a generalised LEM is taken to give extensional support to an otherwise (possibly) unprovable universal quantification that problems arise. It seems likely that this is also the sort of reasoning that led Russell to make the assumption ‘$(\forall\varnothing)$’ in $(***)$ above, and to make it explicitly.

Our own version of Russell’s misgivings, then, might read thus: section 6 shows: that you can prove the LEM inductively (or otherwise) for a property does not mean that you can prove the LEM by induction (or otherwise) for all properties. Indeed, we agree with Russell that whether any given property holds across an infinite domain is certainly not obvious at first glance.

13. *Comparison with Recursive Mathematics*

Recursive mathematics is limited by what sets and functions can be obtained recursively using the natural numbers, and is generally developed using classical logic. Our recursion is specifically directed to the LEM, to try and ensure that it holds. However, we have already pointed out the difficulty in establishing the LEM as a disjunction by mathematical induction, but clearly we can prove the LEM from one of its disjuncts, proved by induction or by any method for that matter. For us, infinite functions and sets can still be defined, whether recursive or not, and, as stated above, their properties are generally determined by meaning rather than by extensionality.

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