ABSTRACT

We construct a generalised reward matrix $Z(s)$, which is an extension of the fluid generator $Q(s)$ of a stochastic fluid model (SFM). We classify the generators that are projections of $Z(s)$, including the generator $Q(s)$, and discuss the application of the resulting generators in different contexts.

As one application example, for the case with nonzero mean drift, we derive a new Riccati equation for the key matrix $\Psi$, which records the probabilities of the first return to the original level.

The Riccati equation has the form $\Psi + \Psi M_{-+} \Psi = M_{++}$, where parameters $M_{++}$ and $M_{-+}$ are block matrices in the matrix $M$, which records the expected number of visits to the original level, before the unbounded fluid drifts to $\pm \infty$.

Finally, we derive the explicit form $\Psi = M_{+-}^{-1} (I + M_{--})^{-1}$.

1. INTRODUCTION

Consider the stochastic fluid model (SFM) [2, 3, 4, 10, 11, 12, 13, 16], denoted $\{(\varphi(t), X(t)) : t \geq 0\}$, with phase variable $\varphi(t) \in S = \{1, \ldots, n\}$ and level variable $X(t) \geq 0$ with a lower boundary at zero, such that:

- $\{\varphi(t) : t \geq 0\}$ is an irreducible continuous-time Markov chain (CTMC) with state space $S$ and generator $T = \{T_{ij}\}_{i,j \in S};$
- $X(t)$ changes at rate $c_{\varphi(t)} = dX(t)/dt$ at time $t$ whenever $X(t) > 0$, and at rate max$\{c_{\varphi(t)}, 0\}$ whenever $X(t) = 0$.

These models have been used in the analysis of a variety of real-life situations, including telecommunications systems [16], risk assessment [6], power generation systems [9] and congestion control [15]. A classical application example is making a telecommunications buffer, using phase $\varphi(t)$ to represent a switch active at time $t$, and the fluid level $X(t)$ to represent the amount of data in the buffer at time $t$. The rate of change $c_i$ of the level in the buffer will depend on whether the switch $i$ that is active at time $t$, lets the data into or out of the buffer, which corresponds to a positive or negative rate, respectively.

The analytical expressions for the stationary and transient analysis of this model have been derived in the literature, and powerful algorithms exist for the numerical evaluations of various performance measures [2, 3, 4, 5, 10, 11, 12, 17].

A key matrix in the theory of SFMs is the fluid generator $Q(s)$ introduced in [11], where $s$ is some complex number. This matrix appears in the expressions for a variety of quantities, including the probability matrices $\Psi$ and $\Xi$ defined in [11].

In this paper, we consider the following generalisation of $Q(s)$. Suppose that while in phase $i$, a reward is accumulated at some constant real-valued rate $r_i$ per unit of time spent in $i$. Further, suppose that we wish to track the accumulation of the reward for different phases individually. In order to model this situation, we construct the generalised reward generator, denoted $Z(s)$, where $s$ is some complex vector. The details of the construction are given in Section 2.

We note that $Q(s)$ is a one-dimensional projection of the multi-dimensional LST generator $Z(s)$, corresponding to the case when the reward is simply time, with $r_i = 1$ for all $i$. The details of this and other projections of $Z(s)$ are given in Section 3.

In Section 4 we classify the generators that are derived from $Z(s)$, and discuss their physical interpretations and applications.

In Section 5, using a particular projection of $Z(s)$, denoted $Z^+(s)$, we derive a new Riccati equation for $\Psi$ and an explicit form of $\Psi$ with the parameters, for both, being derived from the blocks of $Z^+(0)$. Conclusions follow in Section 6.

2. GENERATOR $Z(s)$

In this section we introduce the matrix $Z(s)$ and derive the results that describe its physical interpretation as a generator of the SFM. Matrix $Z(s)$ is a generalisation of the fluid generators $Q(s)$ introduced in [11] and $W(s)$ introduced in [8].

The set $S$ and generator $T$ are partitioned depending on the sign of the rates $c_i$. That is, we partition the set $S$ as $S = S_+ \cup S_0 \cup S_-$, where $S_+ = \{i \in S : c_i > 0\}$, $S_0 = \{i \in S : c_i = 0\}$, and $S_- = \{i \in S : c_i < 0\}$. Further, we partition...
the generator $T$ according to the partition of $S$, as

$$
T = \begin{bmatrix}
S_+ & S_- & S_0 \\
T_{1+} & T_{1-} & T_{10} \\
T_{0+} & T_{0-} & T_{00}
\end{bmatrix},
$$

(1)

Also, we define diagonal matrices $C_+ = \text{diag}(c_i)_{i\in S_+}$ and $C_- = \text{diag}(c_i)_{i\in S_-}$. This notation has been adopted since the various quantities calculated in the analysis of the SFMs appear in a similar block matrix form.

Let $I(\cdot)$ denote the indicator function. For any $i \in S$, let $r_i \in \mathbb{R}$ be some fixed real constant, and define the random variable $W_i(z, t)$ such that

$$
W_i(z, t) = \int_{u=z}^{t} r_i I(\varphi(u) = i) \, du,
$$

(2)

which we interpret as the total $i$-type reward earned during the time interval $[z, t]$, assuming the reward is earned at a rate $r_i$ per unit time spent in $i$.

Also, define diagonal matrices to collect the rates $r_i$, $R_0 = \text{diag}(r_i)_{i \in S_0}$, $R_1 = \text{diag}(r_i)_{i \in S_+}$, and $R_2 = \text{diag}(r_i)_{i \in S_-}$.

Let $\mathbf{s} = [s_i]$ be a (row) vector with $s_i \in \mathbb{C}$, and denote $D_+ = \text{diag}(s_i)_{i \in S_+}$, $D_- = \text{diag}(s_i)_{i \in S_-}$, and $D_0 = \text{diag}(s_i)_{i \in S_0}$.

Assume that $s_1, s_2, \ldots, s_n$ are such that

$$
\chi(A(T_{00} - D_0 R_0)) < 0,
$$

(3)

where $\chi(A)$ denotes the eigenvalue with maximum real part of the matrix $A$. This condition guarantees that the integral $\int_{0}^{t} e^{(T_{00} - D_0 R_0)s} \, ds$ is well defined and then equal to the inverse $(T_{00} - D_0 R_0)^{-1}$. We define the matrix $Z(\mathbf{s})$ as

$$
Z(\mathbf{s}) = \begin{bmatrix}
Z_{++}(\mathbf{s}) & Z_{+-}(\mathbf{s}) & Z_{-+}(\mathbf{s}) & Z_{--}(\mathbf{s})
\end{bmatrix},
$$

(4)

where

$$
Z_{++}(\mathbf{s}) = C_+^{-1}[T_{++} - D_+ R_1 - T_{10}(T_{00} - D_0 R_0)^{-1} T_{0+}],
$$

$$
Z_{+-}(\mathbf{s}) = C_+^{-1}[T_{+-} - D_+ R_- - T_{00}(T_{00} - D_0 R_0)^{-1} T_{0-}-],
$$

$$
Z_{-+}(\mathbf{s}) = C_+^{-1}[T_{-+} - T_{+0}(T_{00} - D_0 R_0)^{-1} T_{0+}],
$$

$$
Z_{--}(\mathbf{s}) = C_+^{-1}[T_{--} - T_{-0}(T_{00} - D_0 R_0)^{-1} T_{0-}].
$$

(5)

Now, define the random variable

$$
h(t) = \int_{u=0}^{t} |\varphi(u)| \, du,
$$

(6)

interpreted as the total amount of fluid that has entered or left the buffer $X(\cdot)$ during the time interval $[0, t]$, and referred to as the in-out fluid [11] of the process $X(\cdot)$. Also, define the random variable, for $y > 0$,

$$
\omega(y) = \inf\{ t > 0 : h(t) = y \},
$$

(7)

interpreted as the first time at which the in-out fluid of the process $X(\cdot)$ reaches level $y$.

Next, as a generalisation of a similar quantity in [8], for any $i, j \in S_+ \cup S_-$, any $y > 0$, $t > 0$, and $w_1, w_2, \ldots, w_n \geq 0$, we define

$$
\delta_i^j(t; w_1, w_2, \ldots, w_n) = P(\varphi(\omega(y)) = j, \omega(y) \leq t, W_k(0, \omega(y)) \leq w_k, k = 1, \ldots, n | X(0) = 0, \varphi(0) = i),
$$

(8)

which we interpret as the joint probability mass/distribution function that, given the process $(\varphi(t), X(t)) : t \geq 0$ starts from level 0 in phase $i$, the in-out fluid of the process $X(\cdot)$ reaches level $y$ for the first time at the time $\omega(y) \leq t$, does so in phase $\varphi(\omega(y)) = j$, and the k-type rewards at time $\omega(y)$ satisfy $W_k(0, \omega(y)) \leq w_k$ for all $k = 1, \ldots, n$.

Also, define the corresponding multi-dimensional Laplace-Stieljes transform (LST) matrix $\mathbf{A}^y(\mathbf{s})$ such that, for any $y > 0$, any vector $\mathbf{s} = [s_i]$ satisfying condition (3), and any $i, j \in S_+ \cup S_-$,

$$
[\mathbf{A}^y(\mathbf{s})]_{ij} = E(e^{-(s_1 W_1(0, \omega(y)) + \cdots + s_n W_n(0, \omega(y)))} \times I(\varphi(\omega(y)) = j, | \varphi(0) = i)),
$$

(9)

$$
= \int_{t=0}^{\infty} \int_{w_1=0}^{\infty} \cdots \int_{w_n=0}^{\infty} e^{-(s_1 w_1 + \cdots + s_n w_n)} \times d\delta^y(i, j; t, w_1, w_2, \ldots, w_n).
$$

(10)

is the LST of the distribution of $(W_1(0, \omega(y)), \ldots, W_n(0, \omega(y)))$ and $\varphi(\omega(y)) = j$, given $\varphi(0) = i$.

Further, for $i, j \in S_0$, $t > 0$, and $w_1, w_2, \ldots, w_n \geq 0$, let

$$
\beta_i^j(t; w_1, w_2, \ldots, w_n) = P(\varphi(t) = j, \varphi(u) \in S_0, 0 \leq u \leq t, W_k(0,t) \leq w_k, k = 1, 2, \ldots, n | \varphi(0) = i),
$$

(11)

which we interpret as the joint probability mass/distribution function that the phase remains in the set $S_0$ at least for the duration of time $t$, the phase at time $t$ is $\varphi(t) = j$, and the k-type rewards at time $t$ satisfy $W_k(0,t) \leq w_k$ for all $k = 1, \ldots, n$, given the process $(\varphi(t), X(t)) : t \geq 0$ starts from level 0 in phase $i$.

Also, define matrix $\mathbf{B}^y(\mathbf{s})$ such that

$$
[\mathbf{B}^y(\mathbf{s})]_{ij} = \int_{w_1=0}^{\infty} \cdots \int_{w_n=0}^{\infty} e^{-(s_1 w_1 + \cdots + s_n w_n)} \times d\beta^y(i, j; t, w_1, w_2, \ldots, w_n),
$$

(12)

is the LST of the distribution of $(W_1(0,t), \ldots, W_n(0,t))$, $\varphi(t) = j$ and the phase process remains in the set $S_0$ at least for the duration of time $t$, given $\varphi(0) = i$.

Theorem 1 proves that the matrix $(T_{00} - D_0 R_0)$ is the generator of $\mathbf{B}^y(\mathbf{s})$.

**Theorem 1.** $\mathbf{B}^y(\mathbf{s})$ is well defined for all $t > 0$, and all $s_i \in \mathbb{C}$, and

$$
\mathbf{B}^y(\mathbf{s}) = e^{(T_{00} - D_0 R_0)t}.
$$

(13)

**Proof:** To prove this result we closely follow the methodology developed in [8, Theorem 1]. The difference is that here, we track the i-type rewards of the possible states $i \in S$ separately.

First, note for all $a, b \geq 0$, and $k = 1, \ldots, n$, that

$$
W_k(0, a + b) = W_k(0, a) + W_k(a, a + b),
$$

and that by conditioning on the state of all the random variables at time $a$ the behaviour of the process during the interval $[0, a)$ is independent of the behaviour of the process during the interval $(a, a + b]$. Therefore, by the law of total
probabilty,
\[
\hat{B}^{(a+b)}(s)_{ij} = \int_{u_1=0}^{r_1(a+b)} \cdots \int_{u_n=0}^{r_n(a+b)} e^{-(s_1 u_1 + \cdots + s_n u_n)} 
\times d\beta^{(a+b)}(j; u_1, \ldots, u_n) 
\times d\beta^{(a)}(k; u_1, \ldots, u_n) 
\times d\beta^{(b)}(h_1; h_1, \ldots, h_n) 
\times d\beta^{(b)}(j; h_1, \ldots, h_n) 
\times d\beta^{(b)}(k; h_n, 0) 
\times d\beta^{(b)}(j; 0, u) 
\times [\hat{B}^{(a)}(s)_{uj}][\hat{B}^{(b)}(s)_{kj}],
\]
and so
\[
\hat{B}^{(a+b)}(s) = \hat{B}^{(a)}(s) \hat{B}^{(b)}(s). \tag{14}
\]
Also, denoting \( \hat{B}^0(s) = \lim_{t \to 0^+} \hat{B}^t(s) \), we have
\[
\hat{B}^0(s) = I, \quad \lim_{y \to 0^+} ||\hat{B}^y(s) - I|| = 0. \tag{15}
\]
By (14) and (15), \( \{\hat{B}^t(s), t > 0\} \) is a strongly continuous semigroup and so \( \hat{B}^t(s) \) must be of the form \( e^{V(s)t} \), where
\[
V(s) = \frac{d}{dh} \hat{B}^h(s) \bigg|_{h=0^+} . \tag{16}
\]
Consider the case when the process starts in phase \( i \in S_0 \) at time zero and is observed at some time \( h > 0 \) in phase \( j \in S_0 \). Given \( h \) is small, there are only two possible events that could occur with probability greater than \( o(h) \).

1. The phase process remains in phase \( i \) until time \( h \). This event occurs with probability \( e^{-\lambda_i h} \) with \( \lambda_i = -T_{ii} \). The corresponding \( i \)-type reward is \( r_i h \), \( k \)-type reward for \( k \neq i \) is zero, and so the LST of the rewards conditional on this event occurring is \( e^u(r_i h) \). We multiply the probability by the conditional LST, and store the result in a diagonal matrix \( \hat{B}^i(s) \) with the \( (i,j) \)-th entry given by
\[
[\hat{B}^i(s)]_{ii} = e^{-(s_i i + \lambda_i) h}.
\]
It follows that,
\[
\frac{d}{dh} [\hat{B}^i(s)]_{ii} \bigg|_{h=0^+} = -(s_i r_i + \lambda_i),
\]
and so
\[
\frac{d}{dh} \hat{B}^i(s) \bigg|_{h=0^+} = -(D_0 R_0 + \Lambda_0), \tag{17}
\]
where \( \Lambda_0 \) is a diagonal matrix with \( \Lambda_0_{ii} = \lambda_i \) for all \( i \in S_0 \).

2. The phase process makes a single transition from \( i \) to phase \( j \neq i \in S_0 \) at some time \( u \in (0, h] \) and remains there until time \( h \). The process undergoes the following set of steps:

* First, the process leaves phase \( i \) at some time \( u \in (0, h] \) and does so with probability density \( \lambda_i e^{-\lambda_i u} \).
* Further, the process makes a transition from phase \( i \) to phase \( j \) with probability \( T_{ij} / \lambda_i \).
* Further, the process remains in phase \( j \) for the remaining \((h-u)\) time with probability \( e^{-\lambda_j (h-u)} \).

Therefore, the probability of this type of event occurring is \( \lambda_i e^{-\lambda_i u} (T_{ij} / \lambda_i) e^{-\lambda_j (h-u)} \). The corresponding \( i \)-type reward is \( r_i u \), \( j \)-type reward is \( r_j (h-u) \), \( k \)-type reward for \( k \neq i,j \) is zero, and so the LST of the rewards conditional on this event occurring is \( e^{-u r_i u - s_j r_j (h-u)} \). We multiply the probability by the conditional LST and integrate over all \( u \in (0, h] \), before finally storing the result in a matrix \( \hat{B}^j(s) \) with the \( (i,j) \)-th entry, for \( i \neq j \), given by
\[
[\hat{B}^j(s)]_{ij} = \int_{u=0}^{h} e^{-u r_i u - s_j r_j (h-u)} \lambda_i e^{-\lambda_i u} T_{ij} e^{-\lambda_j (h-u)} du.
\]
It follows that,
\[
\frac{d}{dh} [\hat{B}^j(s)]_{ij} \bigg|_{h=0^+} = \frac{d}{dh} \left[ T_{ij} e^{-h s_j r_j + \lambda_j} \int_{u=0}^{h} e^{-u (\lambda_i + s_i r_i - \lambda_j - s_j r_j)} du \right]_{h=0^+} = T_{ij},
\]
and so
\[
\frac{d}{dh} \hat{B}^j(s) \bigg|_{h=0^+} = T_{00} + A_0 + \Lambda_0. \tag{19}
\]
Consequently, for all \( i, j \in S_0 \), we have
\[
\left[ V(s) \right]_{ij} = \left[ \frac{d}{dh} (\hat{B}^i(s) + \hat{B}^j(s)) \right]_{h=0^+} \bigg|_{ij} = T_{00} - D_0 R_0 \big|_{ij}. \tag{20}
\]

The result follows.

The theorem below proves that \( Z(s) \) is the generator of \( \Delta V(s) \).

**Theorem** 2. For any \( y > 0 \), \( \hat{\Delta} V(s) \) exists and
\[
\Delta V(s) = e^{Z(s)} . \tag{21}
\]
**Proof:** To prove this result we closely follow the methodology developed in [8, Theorem 2]. The difference is that here, we track the \( i \)-type rewards of the possible states \( i \in S \) separately.
Suppose that the process starts in level 0 from some phase \( i \in S_+ \cup S_- \) at time 0, and the in-out fluid hits level \( y+v \) for the first time at some time \( \omega(y+v) \), and does so in phase \( j \in S_+ \cup S_- \). In order for this to occur,

* first the in-out fluid of the process \( X(\cdot) \) must hit level \( y \) at time \( \omega(y) \), in some phase \( \ell \in S_+ \cup S_- \), and
* next, the in-out fluid of the process \( X(\cdot) \) must hit level \( y+v \) at time \( \omega(y+v) \), in phase \( j \).

Since \( \omega(y) \leq \omega(y+v) \) for \( y, v > 0 \), it follows by (2) that for all \( k = 1, \ldots, n \),
\[
W_k(0, \omega(y+v)) = W_k(0, \omega(y)) + W_k(\omega(y), \omega(y+v)). \tag{22}
\]
As in the previous proof, by conditioning on the state of all random variables at the time \( \omega(y) \), the behaviour of the
process during the interval $[0, \omega(y))$ is independent of the behaviour of the process during the interval $(\omega(y), \omega(y+v)]$, by the memoryless property of the Markov chain, for all $y, v > 0$. By (9), (22), and the law of total probability,

$$\Delta^y [\mathbb{P}](s)_{ij} = E(e^{-s_x W_1(0, \omega(y+v))} \times \mathbb{I}(\varphi(y+v) = j)|\varphi(0) = i)$$

$$= \int_{t=0}^{\infty} \int_{w_1=0}^{\infty} \int_{w_2=0}^{\infty} \cdots \int_{w_n=0}^{\infty} \times e^{-(s_1 w_1 + \cdots + s_n w_n) t} \Delta^y [\mathbb{P}](j, t; w_1, \ldots, w_n)$$

$$= \sum_{\ell \in S_+} \Delta^y [\mathbb{P}]_{i\ell} \times \mathbb{I}(\varphi(y+v) = j)|\varphi(0) = i)$$

Therefore, for all $y, v > 0$,

$$\Delta^y [\mathbb{P}](s) = \Delta^y [\mathbb{P}] \Delta^y [\mathbb{P}]$$  \hspace{1cm} (23)

Also, denoting $\Delta^0 [\mathbb{P}](s) = \lim_{y \to 0^+} \Delta^y [\mathbb{P}](s)$, when $j = i$,

$$\Delta^0 [\mathbb{P}](s)_{i\ell} = \int_{t=0}^{\infty} \int_{w_1=0}^{\infty} \int_{w_2=0}^{\infty} \cdots \int_{w_n=0}^{\infty} \times e^{-(s_1 w_1 + \cdots + s_n w_n) t} \Delta^y [\mathbb{P}]_{i\ell}$$

and when $j \neq i$,

$$\Delta^0 [\mathbb{P}](s)_{ij} = \int_{t=0}^{\infty} \int_{w_1=0}^{\infty} \int_{w_2=0}^{\infty} \cdots \int_{w_n=0}^{\infty} \times e^{-(s_1 w_1 + \cdots + s_n w_n) t} \Delta^y [\mathbb{P}]_{ij}$$

and so

$$\Delta^0 [\mathbb{P}] = I, \lim_{y \to 0^+} ||\Delta^y [\mathbb{P}] - I|| = 0. \hspace{1cm} (24)$$

By (23) and (24), $\{\Delta^y [\mathbb{P}], y > 0\}$ is a strongly continuous semi-group, and so $\Delta^y [\mathbb{P}] = e^{U(s)y}$, where

$$U(s) = \frac{d}{dh} \Delta^h [\mathbb{P}] \bigg|_{h=0^+}. \hspace{1cm} (25)$$

We consider the case $i, j \in S_+$. The proof for the other cases follow by an analogous argument. We will show that

$$U(s)_{ij} = [Z_{++}(s)]_{ij},$$

Assume the process starts from level $X(0) = 0$ in phase $i \in S_+$, and the in-out fluid hits level $h$, for some small $h > 0$, at time $\omega(h)$, and does so in phase $j \in S_+$. Since $h$ is small, there are only three possible events that could occur with a probability greater than $o(h)$.

1. The process remains in phase $i \in S_+$ until the in-out fluid reaches level $h$. In this case, $\omega(h) = h/|c_i|$ and the probability of this occurring is $e^{-\lambda_i(h/|c_i|)}$. The corresponding $i$-type reward is $r_i h/|c_i|$, $k$-type reward for $k \neq i$ is zero, and so the LST of the rewards conditional on this event occurring is $e^{-s_i(r_i h/|c_i|)}$. We multiply the probability by the conditional LST, the result of which is stored in a diagonal matrix $\Delta^h_i$ with the $(i, i)$-th entry given by

$$[\Delta^h_i](s)_{ii} = e^{-s_i(r_i h/|c_i|)} e^{-\lambda_i(h/|c_i|)}$$

It follows that

$$\frac{d}{dh} [\Delta^h_i](s)_{ii} \bigg|_{h=0^+} = -\frac{s_i r_i + \lambda_i}{\Big|c_i\Big|},$$

and so

$$\frac{d}{dh} [\Delta^h_i](s)_{ii} \bigg|_{h=0^+} = -C_i^{-1}(D_i R_i + \Lambda_+), \hspace{1cm} (26)$$

where $\Lambda_+ = \text{diag}(\lambda_i)_{i \in S_+}$.

2. The process makes a single transition from state $i$ to state $j \neq i \in S_+$ when the in-out fluid hits some level $u \in (0, h]$ at time $u/|c_i|$ and remains there until time $\omega(h)$, that is for a further $(h - u)/|c_i|$. Therefore, the probability of this type of event occurring is

$$(\frac{\lambda_i}{|c_i|}) e^{-\lambda_i(u/|c_i|)T_{ii}} e^{-\lambda_j((h-u)/|c_i|)} \hspace{1cm} (27)$$

It follows that $\omega(h) = r_i u/|c_i| + r_j (h - u)/|c_j|$, and the corresponding LST of the rewards conditional on this event occurring is

$$e^{-s_i r_i u/|c_i| + s_j r_j (h-u)/|c_j|} \hspace{1cm} (28)$$

By multiplying (27) and (28), and integrating over all $u \in (0, h]$, we obtain the $(i, j)$th entry of $\Delta^h_i$, given by

$$[\Delta^h_i](s)_{ij} = \int_{u=0}^{h} e^{-s_i r_i u/|c_i| - s_j r_j (h-u)/|c_j|} \left(\frac{1}{|c_i|}\right) \times e^{-\lambda_i(u/|c_i|)T_{ii}} e^{-\lambda_j((h-u)/|c_i|)} du. \hspace{1cm} (29)$$

Therefore, for $i \neq j$,

$$\frac{d}{dh} [\Delta^h_i](s)_{ij} \bigg|_{h=0^+} = \frac{d}{dh} \int_{u=0}^{h} e^{-s_i r_i u/|c_i| - s_j r_j (h-u)/|c_j|} \left(\frac{1}{|c_i|}\right)$$

$$\times e^{-\lambda_i(u/|c_i|)T_{ii}} e^{-\lambda_j((h-u)/|c_i|)} du \bigg|_{h=0^+} \hspace{1cm} (30)$$

and so

$$\frac{d}{dh} [\Delta^h_i](s)_{ij} \bigg|_{h=0^+} = C_i^{-1}(T_{++} + \Lambda_+). \hspace{1cm} (31)$$

3. The process transitions from state $i$ into some $\ell \in S_0$, and then, after spending some time $t$ in $S_0$, transitions into $j \in S_+$, and remains there until time $\omega(h)$.

- First, the process leaves phase $i$ when the in-out fluid hits level $u$ at time $u/|c_i|$, with probability density $(\lambda_i/|c_i|) e^{-\lambda_i(u/|c_i|)}$. 


Next, the process makes a transition from $i \in S_+$ to $\ell \in S_0$ with probability $|d_{i\ell}|/\lambda_i$.

Finally, the process remains in phase $j \in S_+$ until the in-out fluid reaches level $h$, with probability $e^{-\lambda_j((h-s)/|c_j|)}$.

It follows that $\omega(h) = u/|c_i| + t + (h-u)/|c_j|$, the probability of this occurring is

$$\frac{1}{|c_i|} e^{-\lambda_i((h-s)/|c_i|)} |T_{0+} e^{T_{00}^* T_{00}}| |T_{0+}^* e^{T_{00}^* T_{00}}| e^{-\lambda_j((h-s)/|c_j|)}, \quad (32)$$

and the corresponding LST components of the rewards conditional on this event occurring is

$$e^{-s|c_i|}, \quad e^{D_{0} R_{00}}, \quad e^{-(h-s)|c_i|}. \quad (33)$$

By multiplying the terms in (32) and (33) in an appropriate order, and integrating over all $u \in (0,h]$, and $t \in (0,\infty)$, we obtain the $(i,j)$-th entry of $\Delta_3$, given by

$$[\Delta_3^b(s)]_{ij} = \int_{u=0}^{h} \frac{1}{|c_i|} e^{-s|c_i| + s|c_j|} e^{-\lambda_i((h-s)/|c_i|)} \times [T_{0+} e^{T_{00}^* T_{00}} |T_{0+}^* e^{T_{00}^* T_{00}}|]_{ij} \times e^{-\lambda_j((h-s)/|c_j|)} \, du. \quad (34)$$

Since $s$ satisfies the condition (3), the inner integral exists and is given by $-|T_{00} - D_0 R_0|^{-1}$, and so

$$\frac{d}{dh} [\Delta_3^b(s)]_{ij} = \left. \frac{1}{|c_i|} |T_{0+} e^{T_{00}^* T_{00}}| |T_{0+}^* e^{T_{00}^* T_{00}}|^{-1} T_{0+} | \right|_{h=0^+},$$

and

$$\frac{d}{dh} \Delta_3^b(s) \bigg|_{h=0^+} = -C_{i+1} T_{0+} (T_{00} - D_0 R_0)^{-1} T_{0+}. \quad (35)$$

By above,

$$[U(s)]_{ij} = \left[ \frac{d}{dh} [\Delta_3^b(s)]_{h=0^+} \right]_{ij},$$

$$= \left[ \frac{d}{dh} \Delta_3^b(s) + \Delta_3^b(s) \right]_{h=0^+} \bigg|_{ij},$$

$$= [C_{i+1} \{ (T_{i+} - D_0 R_0 - T_{0+} (T_{00} - D_0 R_0)^{-1} T_{0+} ) \}]_{ij},$$

$$= [Z_{++}(s)]_{ij}. \quad (36)$$

In a manner analogous to the argument above, we prove the expression for $Z_{-+}(s)$ (which clearly must have a zero contribution from Case 1). By symmetry, the expressions for $Z_{+-}(s)$ and $Z_{--}(s)$ also follow.

3. PROJECTIONS OF $Z(s)$

Below, we study a range of projections of the generalised reward matrix $Z(s)$ to generators of one-dimensional LSTs, and discuss their applications.

To do this, we define the random variable

$$W(z,t) = \sum_{i \in S} W_i(z,t), \quad (37)$$

interpreted as the total reward earned in all states during the time interval $[z,t]$, and replace $e^{-s|W_i(z,t)|} I(\varphi(\omega(y)) = j)$ with a function of the scalar $s \in \mathbb{C}$, $e^{-s|W(z,t)|}$, in the definitions of the multi-dimensional LSTs. The resulting projections are generators of the one-dimensional LSTs of the distribution of the total reward (37). For example, we replace (9) with

$$|\Delta^b(s)|_{ij} = E(e^{-s|W(0,\omega(y))|} I(\varphi(\omega(y)) = j) \mid \varphi(0) = i), \quad (38)$$

and make associated changes accordingly.

The first example is the fluid generator $Q(s)$ defined in [11],

$$Q(s) = \begin{bmatrix} Q_{++}(s) & Q_{+-}(s) \\ Q_{-+}(s) & Q_{--}(s) \end{bmatrix}, \quad (39)$$

where

$$Q_{++}(s) = C_{++}^{-1} [T_{++} - s I - T_{0+} (T_{00} - s I)^{-1} T_{0+}],$$

$$Q_{-+}(s) = C_{-+}^{-1} [T_{-+} - s I - T_{0-} (T_{00} - s I)^{-1} T_{0-}],$$

$$Q_{+-}(s) = C_{+-}^{-1} [T_{+-} - s I - T_{0+} (T_{00} - s I)^{-1} T_{0-}],$$

$$Q_{--}(s) = C_{--}^{-1} [T_{--} - s I - T_{0-} (T_{00} - s I)^{-1} T_{0+}]. \quad (40)$$

This generator is a projection of $Z(s)$ obtained by setting all reward rates in $r_i = 1$, so that the amount of the reward earned in $i$ is equal to the time spent in $i$.

The physical interpretation of $Q(s)$ established in [11] is that $[e^{Q(i,s)}]_{ij}$ records the LST of the distribution of time for the process to first reach in-out fluid level $y$ and do so in phase $j$, assuming the process starts from level 0 in phase $i$.

The generator $Q(s)$ was used in [10, 11, 12] to evaluate the key matrices $\Psi(s), \Xi(s)$, and related quantities of the SFM $\{(\varphi(t), X(t)) : t \geq 0\}$.

A projection of $Z(s)$ with any real reward rates $r_i$ is the generator $W(s)$ derived in [8],

$$W(s) = \begin{bmatrix} W_{++}(s) & W_{+-}(s) \\ W_{-+}(s) & W_{--}(s) \end{bmatrix},$$

with

$$W_{++}(s) = C_{++}^{-1} [T_{++} - s R_+ - T_{0+} (T_{00} - s R_0)^{-1} T_{0+}],$$

$$W_{-+}(s) = C_{-+}^{-1} [T_{-+} - s R_+ - T_{0-} (T_{00} - s R_0)^{-1} T_{0-}],$$

$$W_{+-}(s) = C_{+-}^{-1} [T_{+-} - s R_+ - T_{0+} (T_{00} - s R_0)^{-1} T_{0-}],$$

$$W_{--}(s) = C_{--}^{-1} [T_{--} - s R_+ - T_{0-} (T_{00} - s R_0)^{-1} T_{0+}]. \quad (41)$$

Generator $W(s)$ was used to analyse the coupled evolution of two fluids: the fluid $\{(\varphi(t), X(t)) : t \geq 0\}$ with a lower boundary zero, as defined in Section 1, and the unbounded fluid $\{(\varphi(t), Y(t)) : t \geq 0\}$ with rates $r_i$. As shown in [8], $[e^{W(s)}]_{ij}$ is the LST of the distribution of the total shift in the fluid $Y(\cdot)$, expressed as $Y(\omega(y)) = Y(0)$, accumulated at the time $\omega(y)$ when the in-out fluid of the process $X(\cdot)$ first reaches level $y$, and $\varphi(\omega(y)) = j$, given $\varphi(0) = i$. In this sense, the amount of the reward earned in $i$ is equal to the shift in $Y(\cdot)$ accumulated while in $i$.

In this paper, of particular interest is the following projection of $Z(s)$, denoted $Z^+(s)$. For a stochastic fluid model $\{(\varphi(t), X(t)) : t \geq 0\}$ with level variable $X(t)$ unbounded
interpreted as the total amount of fluid that has flowed into the buffer $X(t)$ during the time interval $[0, t]$, referred to as the upward shift, since it records the total shift in the fluid during the times the fluid level was increasing.

Suppose that we want to track only this upward shift in the fluid \{$(\varphi(t), X(t)) : t \geq 0$\}. To achieve this, we consider the projection with $W(0, t) = h_+(t)$, let $R_+ = C_+ = R_0 = 0$, and $R_0 = 0$ in (4)-(5), resulting in the matrix

$$Z^+(s) = \begin{bmatrix} Z^+_{++}(s) & Z^+_{+-}(s) \\ Z^+_{-+}(s) & Z^+_{--}(s) \end{bmatrix},$$

(43)

where $Z^+_{++}(s) = Q_{++}(0) - sI$, $Z^+_{+-}(s) = Q_{+-}(0)$, $Z^+_{-+}(s) = Q_{-+}(0)$, and $Z^+_{--}(s) = Q_{--}(0)$.

Here, we establish the following practical implication of $Z^+(s)$. By Theorem 2, for any $i, j \in S$ and $y > 0$,

$$[e^{z(s)\varphi_i}]_{i,j} = E(e^{-s\varphi_1 + (y)}) I(\varphi(y) = j) \mid \varphi(0) = i)$$

(44)

is the LST of the distribution of the total upward shift in $X(\cdot)$ accumulated by the time the in-out fluid of the process $X(\cdot)$ first reaches level $y$ and does so in phase $j$, given that the process starts at phase $i$ at time zero.

In [16], $Z^+(s)$ is used to calculate various loss rates for a doubly-bounded SSM, corresponding to the fluid lost during periods of congestion when the buffer collecting the fluid is full.

By symmetry, for a stochastic fluid model \{$(\varphi(t), X(t)) : t \geq 0$\} with level variable $X(t)$ unbounded below, we define

$$h_-(t) = \int_0^t c_{\varphi(u)} I(i \in S_+) du,$$

(45)

interpreted as the total amount of fluid that has flowed out of the buffer $X(t)$ during the time interval $[0, t]$, referred to as the downward shift, since it records the total shift in the fluid during the times the fluid level was decreasing.

In order to track the downward shift in the fluid, we define matrix $Z^-$, given by

$$Z^-(s) = \begin{bmatrix} Z^-_{++}(s) & Z^-_{+-}(s) \\ Z^-_{-+}(s) & Z^-_{--}(s) \end{bmatrix},$$

(46)

where $Z^-_{++}(s) = Q_{++}(0)$, $Z^-_{+-}(s) = Q_{+-}(0) - sI$, $Z^-_{-+}(s) = Q_{-+}(0)$, and $Z^-_{--}(s) = Q_{--}(0)$.

By Theorem 2, for any $i, j \in S$,

$$[e^{z(s)\varphi_i}]_{i,j} = E(e^{-s\varphi_1 + (y)}) I(\varphi(y) = j) \mid \varphi(0) = i)$$

(47)

is the LST of the distribution of the total downward shift in $X(\cdot)$ accumulated by the time the in-out fluid of the process $X(\cdot)$ first reaches level $y$ and does so in phase $j$, given that the process starts at phase $i$ at time zero.

Note that, in a stochastic fluid model \{$(\varphi(t), X(t)) : t \geq 0$\} with unbounded level variable $X(t) \in (-\infty, +\infty)$, we have $h(t) = h_+(t) + h_-(t)$ and $X(t) = X(0) + h_+(t) - h_-(t)$.

4. Generators Derived from $Z(s)$

In this section we discuss useful applications of generators that are expressed in terms of $Z(s)$.

First, consider the following generalisation of matrices $\Psi(s)$, $\Xi(s)$, $G^{(a,y)}(s)$, and $H^{(x,y)}(s)$ discussed in [7, 11], denoted $\Psi(s)$, $\Xi(s)$, $G^{(a,y)}(s)$, and $H^{(x,y)}(s)$, respectively. Let $\theta(x) = \inf\{t > 0 : X(t) = x\}$ be the first time the process hits level $x$.

Matrix $\Psi(s) = [\Psi(s)]_{i,j} \in S_+ \cap S_-$ is such that, for all $i \in S_+$ and $j \in S_-,$

$$\Psi(s)_{i,j} = E(e^{-s\varphi_1 + (y)\theta(0)} I(\varphi(0) = j) \mid \varphi(0) = i, X(0) = x)$$

(48)

is the LST of the distribution of $(W_1(0,0), \ldots, W_n(0,0))$ and $\varphi(0) = j$, given $\varphi(0) = i$ and $X(0) = 0$.

Matrix $\Xi(s) = [\Xi(s)]_{i,j} \in S_+ \cap S_-$ is symmetrical to $\Psi(s)$ for an unbounded fluid, in which $c_{\varphi_1} = dX(t)/dt$ always, so that $X(t) \in (-\infty, +\infty)$. That is, for all $i \in S_+$ and $j \in S_-$, $\Xi(s)_{i,j}$ is defined by the right-hand side of (48).

Note that, with $0$ denoting a vector of zeros of appropriate size, matrices $\Psi(0)$ and $\Xi(0)$ are equivalent to matrices $\Psi$ and $\Xi$, respectively, defined in [11].

Matrix $G^{(a,y)}(s) = [G^{(a,y)}(s)]_{i,j} \in S_+ \cap S_-$ is such that, for all $i, j \in S_+ \cup S_-$ and $0 < x < y$, $G^{(a,y)}(s)_{i,j} = E(e^{-s\varphi_1 + (y)\theta(0)} I(\varphi(0) = j) \mid \varphi(0) = i, X(0) = x)$

(49)

is the LST of the distribution of $(W_1(0,0), \ldots, W_n(0,0))$ and $\varphi(0) = j$ under the taboo $\theta(x) < \theta(0)$, given $\varphi(0) = i$ and $X(0) = x$. Denote $G^{(a,y)}(s) = lim_{s \to 0^+} G^{(a,y)}(s)$.

Conversely, matrix $H^{(x,y)}(s) = [H^{(x,y)}(s)]_{i,j} \in S_+ \cap S_-$ is such that, for all $i, j \in S_+ \cup S_-$ and $0 < x < y$, $H^{(x,y)}(s)_{i,j} = E(e^{-s\varphi_1 + (y)\theta(0)} I(\varphi(0) = j) \mid \varphi(0) = i, X(0) = x)$

(50)

is the LST of the distribution of $(W_1(0,0), \ldots, W_n(0,0))$ and $\varphi(0) = j$ under the taboo $\theta(x) < \theta(0)$, given $\varphi(0) = i$ and $X(0) = x$. Denote $H^{(x,y)}(s) = lim_{s \to 0^+} H^{(x,y)}(s)$.

We can show by using techniques similar to [11] that when $s_i \geq 0$ for all $i \in S$, $\Psi(s)$ is the minimum nonnegative solution of the Riccati equation

$$Z^-_i(s) + \Psi(s)Z^-_i(s)\Psi(s) + Z^+_i(s)\Psi(s) + \Psi(s)Z^-_i(s) = 0,$$

with similar results for $\Xi(s)$, and algorithms in [12] can be used for finding these solutions. Further, we can derive expressions for $G^{(a,y)}(s)$ and $H^{(x,y)}(s)$ using methodology similar to [7].

Now, we introduce generators $J_1(s)$ and $J_2(s)$, which are generalisations of similar quantities in [10, 11, 12].

$$J_1(s) = Z^-_i(s) + Z^+_i(s)\Psi(s),$$

(51)

$$J_2(s) = Z^+_i(s) + Z^-_i(s)\Xi(s).$$

(52)

These are useful in constructing algorithms for the numerical evaluation of $\Psi(s)$ and $\Xi(s)$, as shown in [12].

The physical interpretation of $J_1(s)$ is that, for $i, j \in S_-$,

$$[J_1(s)]_{i,j} = E(e^{-s\varphi_1 + (y)\theta(0)} I(\varphi(0) = j) \mid \varphi(0) = i, X(0) = y)$$

(53)

is the LST of the distribution of $(W_1(0,0), \ldots, W_n(0,0))$ and $\varphi(0) = j$, given $\varphi(0) = i$ and $X(0) = y$.

}
The physical interpretation of $J_3(s)$ follows by symmetry for the unbounded fluid $X(t) \in (-\infty, +\infty)$.

Further, we define generators $J_3(s)$ and $J_4(s)$, which are generalisations of $K(s)$ in [18].

\begin{align*}
J_3(s) &= Z_{++}(s) + \Psi(s)Z_{--}(s), \quad (54) \\
J_4(s) &= Z_{--}(s) + \Xi(s)Z_{++}(s). \quad (55)
\end{align*}

The physical interpretation of $J_3(s)$ is that for $i, j \in S_+$,

$$ [e^{J_3(s)}y]_{ij} = \int_{\gamma_1 = 0}^{\infty} \cdots \int_{\gamma_n = 0}^{\infty} e^{-(s_1 w_1 + \cdots + s_n w_n)} \gamma_1(y; j; w_1, \ldots, w_n) dw_1 \cdots dw_n \quad (56) $$

is the Laplace transform of the density $\gamma_1(y, j; w_1, \ldots, w_n)$ with respect to the rewards, that the process crosses level $y$ in phase $j$, without avoiding level zero, when the rewards are $(w_1, \ldots, w_n)$, and given $\varphi(0) = i$.

The physical interpretation of $J_4(s)$ follows by symmetry for the unbounded fluid $X \in (-\infty, +\infty)$.

We also define generators $J_5(s)$ and $J_6(s)$, which are generalisations of $(Q_{++} + Q_{--}H^{(b, b)}(0))$ used in [14],

\begin{align*}
J_5(s) &= Z_{++}(s) + Z_{++}(s)H^{(b, b)}(0), \quad (57) \\
J_6(s) &= Z_{--}(s) + Z_{--}(s)G^{(b, b)}(0). \quad (58)
\end{align*}

To establish the physical interpretation of $J_5(s)$, define, for a doubly-bounded fluid $X(t) \in [0, \theta]$, 

\begin{align*}
\hat{h}(t) &= \int_{u=0}^{t} |e^{\varphi(u)}| I(X(u) = b)du, \quad (59) \\
\hat{W}_i(z, t) &= \int_{u=z}^{t} r_i I(\varphi(u) = i, X(u) = b)du, \quad (60)
\end{align*}

interpreted as censored in-out fluid and $t$-type rewards, respectively, accumulated only during periods when the buffer is full.

Let $\delta(y) = \inf\{t > 0 : \hat{h}(t) = y\}$ be the first time the censored in-out fluid reaches level $y$. Then, for all $i, j \in S_+$,

$$ [e^{J_5(s)}y]_{ij} = E(e^{-(s_1 \hat{W}_1(0, \delta(y))) + \cdots + s_n \hat{W}_n(0, \delta(y)))} \cdot I(\varphi(\delta(y)) = j, \delta(y) < \theta(0)) | \varphi(0) = i, X(0) = b) \quad (61) $$

is the LST of the distribution of $(\hat{W}_1(0, \delta(y)), \ldots, \hat{W}_n(0, \delta(y)))$ and $\varphi(\delta(y)) = j$ under the taboo $\delta(y) < \theta(0)$, given $\varphi(0) = i$ and $X(0) = b$.

The physical interpretation of $J_6(s)$ follows by symmetry, for the rewards earned only when the buffer is empty.

5. NEW RICCATI EQUATION FOR $\Psi$

In this section, we derive a new Riccati equation for $\Psi$ and an explicit expression for $\Psi$ in terms of appropriately defined matrix $M$. We are currently investigating whether this new matrix $M$ can be computed efficiently.

Consider a stochastic fluid model $\{(\varphi(t), X(t)), t \geq 0\}$ with unbounded level variable $X(t) \in (-\infty, +\infty)$. Throughout this section, we assume that the process is transient.

(Note that the only other alternative is null-recurrence.)

For $0 \leq x \leq y$, define $f_y(x) = \int f_x(y)_{ij} | i, j \in S_+ \cup S_- $ such that, for all $i, j \in S_+ \cup S_-$, $f_y(x)_{ij}$ is the inverse of the LST $[e^{Z^+(s)}y]_{ij}$ so that

$$ f_y(x)_{ij} = \frac{d}{dx} P(h_+(\omega(y)) < x, \varphi(\omega(y)) = j | X(0) = 0, \varphi(0) = i), \quad (62) $$

is the probability density that the total upward shift in $X(t)$ at the time $\omega(y)$ is $h_+(\omega(y)) = x$ and the phase is $\varphi(\omega(y)) = j$, given that the process starts in phase $i$ at time zero. It follows that $\sum_y f_y(x)_{ij} dy = 1$.

Note that by using the method of Abate and Whitt [1], we can obtain $f_y(x)$ by numerically inverting $e^{Z^+(s)y}$.

**Theorem 3.** Let $M = \{M_{ij}\}$ be a matrix defined by

$$ M_{ij} = \int_{y=0}^{\infty} f_y(y/2)_{ij} dy, \quad (63) $$

and partitioned according to $S_+ \cup S_-$ as

$$ M = \begin{bmatrix} M_{++} & M_{+} \\ M_{-} & M_{--} \end{bmatrix}. \quad (64) $$

Then $M$ has the form

$$ M = \begin{bmatrix} \Psi M_{++} & \Xi \Psi^{-1} \Psi M_{+-} \\ \Xi (I - \Psi \Xi)^{-1} \Xi M_{-+} \end{bmatrix}. \quad (65) $$

**Proof:** First, note that by (63), the quantity $M_{ij}$ is the expected number of visits to state $(j, 0)$ given that the process starts in state $(i, 0)$ for all $i, j \in S_+ \cup S_-$. This follows from the facts that:

(i). $f_y(y/2)_{ij}$ is the density that given the process started from level 0 in phase $i$, it will be on level 0 in phase $j$ when the total in-out fluid is $y$ (in which case the total upward shift is $y/2$); and

(ii). integrating the density $f_y(y/2)_{ij}$ gives the expected number of visits to $(j, 0)$ given start in $(i, 0)$, by standard results in probability theory.

Consider $M_{+-} = \{M_{ij}\}_{i, j \in S_+ \cup S_-}$, the analysis for the remaining block matrices is analogous.

Assume $i \in S_+$, $j \in S_-$ and $\varphi(0) = i$, $X(0) = 0$. Denote $\theta_0 = 0$, and let $\theta_n = \inf\{t > \theta_{n-1} : X(t) = 0, \varphi(t) \in S_+\}$ for $n = 1, 2, \ldots$. We interpret $\theta_n$ as the time of the $n$th crossing of level zero from above, for all $n = 0, 1, 2, \ldots$.

Define matrix $P(n) = \{P(n)_{ij}\}_{i, j \in S_+ \cup S_-}$ such that, for $i \in S_+$, $j \in S_-,

$$ P(n)_{ij} = \int_{\omega_n < \infty, \varphi(\omega_n) = j} I(\varphi(0) = i, X(0) = 0) \quad (66) $$

is the probability that the $n$th crossing of level zero from above occurs in finite time and that the process is in phase $j$ at the time of the $n$th crossing of level zero from above, given that the process starts with $\varphi(0) = i$, $X(0) = 0$.

By the standard theory of discrete-time Markov chains,

$$ [M_{+-}]_{ij} = \sum_{n=0}^{\infty} [P(n)]_{ij}. \quad (67) $$

Now, for $n = 0, 1, 2, \ldots$,

$$ [P(n)]_{ij} = [(\Psi \Xi)^n \Psi]_{ij}, \quad (68) $$

and so

$$ [M_{+-}]_{ij} = \sum_{n=0}^{\infty} [(\Psi \Xi)^n \Psi]_{ij} = [(I - \Psi \Xi)^{-1} \Psi]_{ij}, \quad (69) $$
where the inverse \((I - \Psi \Xi)^{-1}\) exists since the process is transient. Therefore,
\[
M_{+} = (I - \Psi \Xi)^{-1} \Psi. \quad (70)
\]
The other block matrices in \(M\) can be constructed in an analogous manner. 

Below, we state new results for \(\Psi\).

**Corollary 1.** \(\Psi\) is a solution to the Riccati equation
\[
M_{+} = \Psi + \Psi M_{-} \Psi. \quad (71)
\]

**Proof:** By (65),
\[
\Xi M_{+} = M_{-} \Psi.
\]
Since the process is transient, we can write
\[
\Psi = (I - \Psi \Xi)(I - \Psi \Xi)^{-1} \Psi = (I - \Psi \Xi)M_{+} = M_{+} - \Psi M_{-} \Psi,
\]
and so the result follows. 

**Corollary 2.** \(\Psi\) can be explicitly written as
\[
\Psi = M_{+}(I + M_{-})^{-1}. \quad (72)
\]

**Proof:** By (65),
\[
M_{+} = (I - \Psi \Xi)^{-1} \Psi,
\]
\[
\Rightarrow M_{+} - \Psi M_{-} = \Psi, \text{ since } M_{-} = \Xi M_{+},
\]
\[
\Rightarrow \Psi = M_{+}(I + M_{-})^{-1}.
\]
To justify the existence of \((I + M_{-})^{-1}\), note that by (65),
\[
M_{-} = \Xi(I - \Psi \Xi)^{-1} \Xi = \sum_{n=1}^{\infty} (\Psi \Xi)^n,
\]
and
\[
I + M_{-} = \sum_{n=0}^{\infty} (\Psi \Xi)^n = (I - \Psi \Xi)^{-1},
\]
for all transient processes.

6. **CONCLUSION**

We have constructed a generalised reward generator \(Z(s)\) for the stochastic fluid model useful for tracking the accumulation of reward for different phases individually.

We have considered various projections of \(Z(s)\), including the fluid generators \(Q(s)\) [11] and \(W(s)\) [8].

We constructed the generator \(Z^+(s)\) which tracks the upward shift in the fluid. We applied \(Z^+(s)\) to construct the matrix \(M\) which records the expected number of visits to the original level before the unbounded fluid drifts off to \(\pm \infty\).

We used the elements of \(M\) and its physical interpretation to derive a new Riccati equation and an explicit solution for the matrix \(\Psi\), which is a key building block to many other performance measures. Work on the algorithmic techniques resulting from this equation is in progress.

7. **REFERENCES**


