AN IDENTITY FOR COCYCLES ON COSET SPACES OF LOCALLY COMPACT GROUPS

H.KUMUDINI DHARMADASA AND WILLIAM MORAN

Abstract. We prove here an identity for cocycles associated with homogeneous spaces in the context of locally compact groups. Mackey introduced cocycles (λ-functions) in his work on representation theory of such groups. For a given locally compact group G and a closed subgroup H of G, with right coset space G/H, a cocycle λ is a real-valued Borel function on $G/H \times G$ satisfying the cocycle identity

$$\lambda(x, st) = \lambda(x.s, t)\lambda(x, s), \quad \text{a.e. } x \in G/H, s, t \in G,$$

where the "almost everywhere" is with respect to a measure whose null sets pull back to Haar measure null sets on G. Let H and K be regularly related closed subgroups of G. Our identity describes a relationship between cocycles for G/H^x, G/K^y and $G/(H^x \cap K^y)$ for almost all $x, y \in G$. This also leads to an identity for modular functions of G and the corresponding subgroups.

1. Introduction and Statement of Results

The aim of this paper is to prove an identity for cocycles (Mackey's λ functions). The need for this identity arose in connection with problems on induced representations to be discussed in a later publication. Let G be a separable locally compact group and H a closed subgroup of G. In his treatment of Induced Representation on Locally Compact Groups ([4]), Mackey introduced the concept of a cocycle λ as a real valued positive function on $(G/H) \times G$ satisfying certain identities (see sec.2.1). Most importantly, such cocycles are associated with quasi-invariant measures; to each such cocycle there is a quasi-invariant measure μ on G/H, so that the Radon-Nikodym derivative of the translated measure $\mu.s$ with respect to μ is $\lambda(\cdot, s)$ for all $s \in G$. Once λ is specified, this measure is unique up to a positive scalar multiple. The basic properties of these functions are well-established in the literature (cf. [1, 4, 5]).

To state our results, we will need some concepts and notations. Let λ_H denote a cocycle corresponding to the homogeneous space G/H and let $H^x = x^{-1}Hx$, for $x \in G$. Let Δ_H be the modular function of the group H. Closed subgroups H and K of G are said to be *regularly related* if the double coset space $H\backslash G/K$ is a standard Borel space (cf.[4]). We note that the double coset space formed by the right action of the diagonal subgroup $\Lambda = \{(x, x) : x \in G\}$ of $G \times G$ on the coset space $(H \times K)\backslash(G \times G)$ (that is, $H \times K\backslash G \times G/\Lambda$) is identified with $H\backslash G/K$ by the map $(x, y) \mapsto xy^{-1}$. The regularly related property for H and K is equivalent to this action being smooth ([4]).

Our main result provides a link between cocycles for conjugates of regularly related subgroups. We will abuse notation and assume that the cocycle $\lambda_H(s, t)$ is actually defined on $G \times G$ and constant on the right cosets of H, rather than on $(G/H) \times G$.

Theorem 1. *Let G be a separable locally compact group and H, K closed subgroups of G. If H and K are regularly related, then for each double coset $D(x, y) = (H \times K)(x, y)\Lambda$, there is a quasi-invariant measure*

Date: April, 24, 2016.
2010 *Mathematics Subject Classification.* 43A65, 22D30, 43A15.
Key words and phrases. Separable locally compact group, quasi-invariant measure, modular function, $\lambda-$ function.

$\mu_{x,y}$ on $G/(H^x \cap K^y)$, $x, y \in G$, and a corresponding cocycle $\lambda_{H^x \cap K^y}$ such that

$$\lambda_{H^x}(ts^{-1}, s)\lambda_{K^y}(ts^{-1}, s)\lambda_{H^x \cap K^y}(t, s^{-1}) = 1, \quad (s, t \in G, a.e.(x, y) \in (G \times G)/(H \times K)). \tag{1}$$

Moreover, $\lambda_{H^x \cap K^y}(t, s)$ is defined everywhere and continuous on $(G/(H^x \cap K^y)) \times G$.

Theorem 1 leads to an identity relating the modular functions corresponding to the subgroups involved:

Corollary 1. *For $(x, y) \in G \times G$ such that (1) holds, and for $s \in H^x \cap K^y$,*

$$\frac{\Delta_{H^x}(s)\Delta_{K^y}(s)}{\Delta_G(s)\Delta_{H^x \cap K^y}(s)} = 1.$$

A necessary and sufficient condition for the existence of an invariant measure on a quotient group G/H of G is that $\Delta_H(x) = \Delta_G(x)$ for all $x \in H$. We say that $G \supset H$ is *comodular* if this happens. Now we have the following straightforward consequence of Corollary 1.

Corollary 2. *Let H and K be regularly related closed subgroups of a separable locally compact group G and $(x, y) \in G \times G$ such that the identity (1) holds. If $G \supset H$ is comodular, then $K \supset (H^z \cap K)$ is comodular for almost all $z \in G$.*

2. Preliminaries on cocycles

To avoid measure theoretic complications, we assume throughout that G is a locally compact separable group and H a closed subgroup. We denote the right-invariant Haar measure on G by ν_G, with e denoting the identity of the group. The canonical mapping from G to the set of right-cosets G/H is denoted by p_H. Throughout this section, X denotes the set of right cosets G/H of H with the standard *right* action of G. A *left* action by any other (closed) subgroup K of G gives rise to orbits in one-to-one correspondence with the double cosets $H\backslash G/K$, and the stabilizer of $Hx \in X$ under the action of K is $H^x \cap K$.

We briefly list the key results on cocycles, quasi-invariant measures and related concepts from our perspective. The reader is referred to [1, 4, 5].

- There is a *regular Borel section* $B \subset G$; that is, a Borel set B that intersects each right G coset in exactly one point such that $(p_H^{-1}(p_H(K))) \cap B$ has a compact closure for each compact subset K of G.
- A strictly positive, real-valued continuous function ρ_H exists on G satisfying

  $$\rho_H(hx) = (\Delta_H(h)/\Delta_G(h))\rho_H(x), \quad (x \in G, h \in H).$$

- Such a *ρ-function* gives rise to a unique Borel cocycle λ_ρ on $X \times G$ such that

  $$\lambda_\rho(p_H(s), y) = \frac{\rho(sy)}{\rho(s)}, \quad (s, y \in G)$$

  with the properties:
  (a) for all $x \in X$ and $s, t \in G$, $\lambda_\rho(x, st) = \lambda_\rho(x.s, t)\lambda_\rho(x, s)$;
  (b) for all $h \in H, \lambda_\rho(p_H(e), h) = \Delta_H(h)/\Delta_G(h)$;
  (c) for $t \in G$, $\lambda_\rho(p_H(e), t)$ is bounded on compact sets as a function of t; For $x, t \in G$ and for almost all $v \in G/H$, $\lambda_{H^x}(x^{-1}v, t) = \lambda_H(v, t)$.
- For each ρ-function on G there is a quasi-invariant measure μ on X such that, for all $y \in G$, the corresponding cocycle λ_ρ has the property that $\lambda_\rho(\cdot, y)$ is a Radon-Nikodym derivative of the translation measure $\mu.y$ with respect to the measure μ.

- For $x \in G$ let $\dot{x} = p_H(x)$. If μ denotes the quasi-invariant measure corresponding to the function ρ then

$$\int_G f(x)\rho(x)d\nu_G(x) = \int_X \int_H f(hx)d\nu_H(h)d\mu(\dot{x}), \quad (f \in C_{00}(G)).$$

where $C_{00}(G)$ denotes the continuous functions on G with compact support. We write $\mu \succ \lambda$ to mean that for all $y \in G$, $\lambda(\cdot, y)$ is the cocycle that is the Radon-Nikodym derivative of the translate $E \mapsto \mu([E]y)$ with respect to μ. The following facts on λ and corresponding ρ-functions can be found in many places in the literature (see, for example, [4, 1]).

There are quasi-invariant measures on X, any two of which are absolutely continuous. Null sets for such measures are exactly those sets E for which $p_H^{-1}(E)$ has Haar measure zero. The relations $\mu \succ \lambda$ and $\lambda = \lambda_\rho$ between quasi-invariant measures, cocycles, and ρ-functions have the following properties:

(i) Every cocycle is of the form λ_ρ; $\lambda_{\rho_1} = \lambda_{\rho_2}$ if and only if ρ_1/ρ_2 is a constant.
(ii) For every cocycle there is a quasi-invariant measure μ such that $\mu \succ \lambda$; if $\mu_1 \succ \lambda$ and $\mu_2 \succ \lambda$ then μ_1 is a constant multiple of μ_2.
(iii) For every quasi-invariant measure μ there is a cocycle λ such that $\mu \succ \lambda$. If $\mu \succ \lambda_1$ and $\mu \succ \lambda_2$ then, for all t, $\lambda_1(\cdot, t) = \lambda_2(\cdot, t)$ a.e.
(iv) If $\mu \succ \lambda_{\rho_1}$ and $\mu \succ \lambda_{\rho_2}$ then ρ_1/ρ_2 is constant almost everywhere.

3. Proofs of the Results

First we recall some standard results on disintegration of measures. Let X be a separable locally compact space supporting a finite measure μ and let R be an equivalence relation on X where $r(x)$ is the equivalence class containing x. The relation R is *measurable* if there exists a countable family $E_1, E_2, \dots$ of subsets of X/R such that $r^{-1}(E_i)$ is measurable for each i and such that each point in X/R is the intersection of the E_i containing it (cf. [4, 2]).

It is well known (see, for example, [4], p.124, Lemma 11.1) that the measure μ is decomposable as an integral over X/R of measures μ_y concentrated on the equivalence classes.

If $\tilde{\mu}$ is the "push-forward" measure on X/R from the measure μ on X, i.e. $\tilde{\mu}(E) = \mu(r^{-1}(E))$, then for each y in X/R there exists a finite Borel measure μ_y on X such that $\mu_y(X - r^{-1}(\{y\})) = 0$ and

$$(2) \qquad \int f(y) \int g(x)d\mu_y(x)d\tilde{\mu}(y) = \int f(r(x))g(x)d\mu(x),$$

whenever $f \in L_1(X/R, \tilde{\mu})$ and g is bounded and measurable on X. If μ is quasi-invariant then in the disintegration of μ in (2) above, μ_y is also quasi-invariant under the action of G a.e. y ([4]).

Proof of Theorem 1.

It is clear that if H and K are regularly related then the orbits of G/H under the action of K outside a set of measure zero form the equivalence classes of a measurable equivalence relation. The right action of the diagonal subgroup $\Lambda = \{(x,x) : x \in G\}$ of $G \times G$ on the coset space $(G \times G)/(H \times K)$ has stabilizer $H^x \times K^y \cap \Lambda = (H \times K)^{(x,y)} \cap \Lambda$ at (Hx, Ky), and the orbit of this point is the double coset $(H \times K)(x,y)\Lambda$. We write Υ for the set of all double cosets $(H \times K)\backslash G \times G/\Lambda \simeq H\backslash G/K$. As noted earlier, the regularly related property for H and K is equivalent to this orbit space being smooth ([4]). Writing $D(x,y)$ for the double coset to which (x,y) belongs, for a fixed finite measure ν_0 on $G \times G$ equivalent to Haar measure, we define a measure $\mu_{(H,K)}$ on Υ by $\mu_{(H,K)}(F) = \nu_0(D^{-1}(F))$. Such a measure is called an *admissible measure* by Mackey.

Fix a finite product measure $\nu_0 = \nu_1 \times \nu_2$ on $(G \times G)$ equivalent to Haar measure. Let $\mu_{H\times K}$ be the image of ν_0 under $p_{H\times K}$ and μ_H, μ_K the images of ν_1, ν_2 under p_H and p_K respectively. Let $\mu_{H,K}$ be an admissible measure in Υ corresponding to ν_0.

For a function f on $(G/H) \times (G/K)$ for which $\int_{G/H}\int_{G/K} f(x,y)d\mu_H(x)d\mu_K(y)$ is integrable, using the change of variables $x \mapsto xs$ and $y \mapsto ys$, we obtain

$$\int_{G/H}\int_{G/K} f(x,y)d\mu_H(x)d\mu_K(y) = \int_{G/H}\int_{G/K} \lambda_H(x,s)\lambda_K(y,s)f(xs,ys)d\mu_H(x)d\mu_K(y)$$
$$= \int_{(G\times G)/(H\times K)} \lambda_H(x,s)\lambda_K(y,s)f(xs,ys)d\mu_{H\times K}(x,y).$$

For $(x,y) \in (G\times G)/(H\times K)$, write $r(x,y) = D(p^{-1}_{H\times K}(x,y))$; this defines a measurable equivalence relation since H and K are regularly related. The measure $\mu_{H\times K}$ is disintegrated into an integral of measures $\mu_{x,y}$, where $D(x,y) \in \Upsilon$, with respect to the measure $\mu_{H,K}$ on Υ. Also, each $\mu_{x,y}$ is a quasi-invariant measure on the orbit $r^{-1}(D(x,y))$ (cf. (2)). Using this disintegration, we have

$$\int_{(G\times G)/(H\times K)} \lambda_H(x,s)\lambda_K(y,s)f(xs,ys)d\mu_{H\times K}(x,y)$$
$$= \int_{D\in\Upsilon}\int_{\underline{t}\in\Lambda/(H\times K)^{(x,y)}\cap\Lambda} \lambda_H(xt,s)\lambda_K(yt,s)f(xts,yts)d\mu_{x,y}(\underline{t})d\mu_{H,K}(D),$$

where (x,y) is a coset representative of the coset $D(x,y)$. Identifying the space $\Lambda/((H\times K)^{(x,y)}\cap\Lambda)$ with $G/(H^x\cap K^y)$, we can regard $\mu_{x,y}$ as a measure on $G/(H^x\cap K^y)$. Then we have

$$\int_{(G\times G)/(H\times K)} \lambda_H(x,s)\lambda_K(y,s)f(xs,ys)d\mu_{H\times K}(x,y)$$
$$= \int_{D\in\Upsilon}\int_{t\in G/(H^x\cap K^y)} \lambda_H(xt,s)\lambda_K(yt,s)f(xts,yts)d\mu_{x,y}(t)d\mu_{H,K}(D). \tag{3}$$

Changing variables $t \mapsto ts^{-1}$ in the integral on the right-hand side, we find that

$$\int_{(G\times G)/(H\times K)} \lambda_H(x,s)\lambda_K(y,s)f(xs,ys)d\mu_{H\times K}(x,y)$$
$$= \int_{D\in\Upsilon}\int_{t\in G/(H^x\cap K^y)} \lambda_H(xts^{-1},s)\lambda_K(yts^{-1},s)f(xt,yt)\lambda_{H^x\cap K^y}(t,s^{-1})d\mu_{x,y}(t)d\mu_{H,K}(D). \tag{4}$$

On the other hand, if we start with $\int_{(G\times G)/(H\times K)} f(x,y)d\mu_{H\times K}(x,y)$ and use disintegration, we have

$$\int\int_{(G\times G)/(H\times K)} f(x,y)d\mu_{H\times K}(x,y)$$
$$= \int_{D\in\Upsilon}\int_{\underline{t}\in\Lambda/(H\times K)^{(x,y)}\cap\Lambda} f(xt,yt)d\mu_{x,y}(t,t)d\mu_{(H,K)}(D)$$
$$= \int_{D\in\Upsilon}\int_{t\in G/(H^x\cap K^y)} f(xt,yt)d\mu_{x,y}(t)d\mu_{(H,K)}(D). \tag{5}$$

Now (4) and (5) yield

$$\lambda_H(xts^{-1},s)\lambda_K(yts^{-1},s)\lambda_{H^x\cap K^y}(t,s^{-1}) = 1, \quad (s\in G, \text{a.e. } t\in G/(H^x\cap K^y)); \tag{6}$$

or, using the cocycle property (c) in Sec.2,

(1) $\lambda_{H^x}(ts^{-1}, s)\lambda_{K^y}(ts^{-1}, s)\lambda_{H^x\cap K^y}(t, s^{-1}) = 1, \quad (s \in G, \text{a.e. } t \in G/(H^x \cap K^y))$

for almost all $(x, y) \in (G \times G)/(H \times K)$. Fixing such an (x, y) in $(G \times G)/(H \times K)$, and invoking continuity of λ_H and λ_K, we see that (1) is true for all $t \in G/(H^x \cap K^y)$. Furthermore, (1) implies that $\lambda_{H^x\cap K^y}(t, s)$ is defined everywhere and continuous on $(G/(H^x \cap K^y)) \times G$.

Proof of Corollary 1. Setting $t = s$ in (1) and using the property (a) of cocycles in Sec. 2.1, we obtain

(7) $\lambda_{H^x}(e, s)\lambda_{K^y}(e, s) = \lambda_{H^x\cap K^y}(e, s).$

Now we use the property (b) of cocycles in Sec 2.1 to obtain

(8) $$\frac{\Delta_{H^x}(s)}{\Delta_G(s)}\frac{\Delta_{K^y}(s)}{\Delta_G(s)} = \frac{\Delta_{H^x\cap K^y}(s)}{\Delta_G(s)}$$

This leads to the required equality

(9) $$\frac{\Delta_{H^x}(s)}{\Delta_G(s)}\frac{\Delta_{K^y}(s)}{\Delta_{H^x\cap K^y}(s)} = 1.$$

Remarks:

- We emphasise that the result is an almost everywhere statement on the product space $G/H \times G/K$. If $H = K$, the diagonal $\{(x, x) : x \in G/H\}$ will normally have zero measure. Indeed, if it has non-zero measure, so that our results allow us to make statements about the comodularity of $G \supset H^x = H^x \cap H^x$, the quotient space G/H is discrete, and so H is an open subgroup. In that case, it is already well-known (and trivial) that $\Delta_G(h) = \Delta_{H^x}(h)$ for all $h \in H^x$ and all x.
- If we consider the special case where $K = e$, we have $K^y = e$ for all $y \in G$, giving $s = e$. The conclusion from Corollary 1 in this case is trivial.
- If H is a normal subgroup of G, then $H^x = H$ for all $x \in G$ and we have $\Delta_H(s) = \Delta_G(s)$ in consequence of the normality. Here, with an application of Fubini's Theorem (cf.[3], p.153), Corollary 1 becomes
$$\Delta_{H^x\cap K^y}(s) = \Delta_{K^y}(s), \text{ for } s \in H \cap K^y.$$
But this is a fact anyway, since $H \cap K^y$ is normal in K^y.

Proof of Corollary 2. If $G \supset H$ is comodular, then so is $G \supset H^x$ for all $x \in G$. An application of Corollary 1 implies that $H^x \supset H^x \cap K^y$ is comodular for almost all x and y. By conjugating with y^{-1} and using Fubini's Theorem, it then follows by conjugation, that $K \supset H^z \cap K$ is comodular for almost all $z \in G$.

References

[1] Gaal, Steven A.: Linear Analysis and Representation Theory. Springer, Berlin, (1973).
[2] Halmos, P.R.: The decomposition of measures. Duke Math. J.,Vol 8 (1941) p.386-392.
[3] Hewitt, E. and Ross, K.A.: Abstract Harmonic Analysis, Vol I, Springer, Berlin, (1963).
[4] Mackey, G.W.: Induced representations of locally compact groups I. Annals of Maths, Vol55, No1, (1952), p. 101-140.
[5] Reiter, Hans: Classical Harmonic Analysis and Locally Compact Groups. Oxford Mathematical Monographs, (1968).

Discipline of Mathematics, School of Physical Sciences, University of Tasmania, Tasmania 7001, Australia.
E-mail address: kumudini@utas.edu.au

Building 12, Level 7, Room 20, School of Electrical and Computer Engineering, RMIT University Melbourne City Campus, Melbourne, Victoria 3001, Australia.
E-mail address: bill.moran@rmit.edu.au