

Born reciprocity and the $1/r$ potential

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Abstract. Many structures in nature are invariant under the transformation pair, $(\mathbf{p}, \mathbf{r}) \rightarrow (b\mathbf{r}, -\mathbf{p}/b)$, where b is some scale factor. Born's reciprocity hypothesis affirms that this invariance extends to the entire Hamiltonian and equations of motion. We investigate this idea for atomic physics and galactic motion, where one is basically dealing with a $1/r$ potential and the observations are very accurate, so as to determine the scale $b \equiv m\Omega$. We find that an $\Omega \sim 1.5 \times 10^{-15} \text{ s}^{-1}$ has essentially no effect on atomic physics but might possibly offer an explanation for galactic rotation, without invoking dark matter.

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1. Born's Reciprocity Principle

One cannot help but be struck by the way that numerous structures in physics look the same under the simultaneous substitution between momentum \mathbf{p} and position \mathbf{r} ,

$$\mathbf{p} \rightarrow b\mathbf{r}, \quad \mathbf{r} \rightarrow -\mathbf{p}/b, \quad (1)$$

where b is a scale with the dimensions of M/T. This applies to the classical Poisson brackets $\{r_i, p_j\} = \delta_{ij}$, the quantum commutator brackets $[r_i, p_j] = i\hbar\delta_{ij}$, and the form of the Hamiltonian equations (classical or quantum), $\dot{r}_i = \partial H/\partial p_i$, $\dot{p}_i = -\partial H/\partial r_i$ and of the angular momenta, $L_{ij} = r_i p_j - r_j p_i$. It leads to the concept of phase-space, Fourier transforms and uncertainty relations. The conjugacy between space and momentum is extensible to energy and time, this being required for special relativity. However these observations do not presume that Hamiltonians are invariant under transformation (1).

Born's reciprocity principle [1, 2] goes one stage further and assumes that all physical equations of motion are *invariant* and not just covariant under such conjugacy transformations, so that

$$H(\mathbf{r}, \mathbf{p}) = H(-\mathbf{p}/b, b\mathbf{r}). \quad (2)$$

At first sight this seems a patently absurd idea for anything but oscillators and even there it seems quite silly because it leads to fixed frequency for all vibrations, determined by the value of b . For these reasons and for its failure in accounting for the observed particle masses [1] the principle has naturally fallen into disrepute and has never been taken seriously by physicists. There are also some fundamental philosophical objections to the idea, which will be mentioned later. In spite of these very valid criticisms, we wish to explore the principle and see if we can determine a non-zero value of b (which is probably tiny indeed and tied to cosmic scales). The incentive/reason why we wish to entertain the chance that eq.(2) may be valid arises also from vibrations. The point is that any oscillator contains some measure of anharmonicity; how this is manifested depends on the physical context, but we can be sure that the linear force and its associated potential V cannot increase without end. For instance we might suppose that in reality, for mass m , the true potential is $V(r) \simeq (m\omega^2 r^2/2) \exp(-cr^2)$, where c is a small parameter which sets the distance at which anharmonicity kicks in, and ω is the natural rotational frequency for displacements that are not excessive. If we attempt to make H reciprocity invariant, we may arrive at

$$2H = (p^2 + b^2 r^2)/m + m\omega^2 [r^2 \exp(-cr^2) + (p^2/b^2) \exp(-cp^2/b^2)].$$

It follows that if b is miniscule on ordinary momentum scales, the dangerous last term is minute, as is the correction to the kinetic energy, so the standard picture prevails for small or moderate r . This example indicates we have no right to be so dismissive of Born's principle.

If we view the reciprocity substitution as the transformation

$$\begin{pmatrix} \mathbf{p} \\ b\mathbf{r} \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{p} \\ b\mathbf{r} \end{pmatrix},$$

we can think of another reciprocity transformation which also leaves most of our physical structures intact, namely

$$\begin{pmatrix} \mathbf{p} \\ b\mathbf{r} \end{pmatrix} \rightarrow \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \begin{pmatrix} \mathbf{p} \\ b\mathbf{r} \end{pmatrix}, \quad \begin{pmatrix} H \\ bt \end{pmatrix} \rightarrow \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} H \\ bt \end{pmatrix}, \quad m \rightarrow im.$$

In this paper we will consider a non-relativistic potential which is fully established on the large-scale (Newtonian gravity) and on the small scale (atomic physics), namely $1/r$; it has its roots in graviton and photon exchange and has been thoroughly studied over the centuries! We shall explore how Born's principle affects it. The immediate question is how to make $V(r) \propto 1/r$ compatible with (1). The substitution, $1/r \rightarrow 1/\sqrt{r^2 + p^2/b^2}$ can be rejected outright as it would enormously enhance the velocity dependence for small b , in fact ridiculously so. However a more reasonable alternative is $1/r \rightarrow (1/r + b/p)$, since the last term fades out as $b \rightarrow 0$. ‡ There may exist other choices for making V compatible with Born's principle, but they are probably less natural than the proposal:

$$H(\mathbf{r}, \mathbf{p}) = \frac{\mathbf{p}^2 + b^2\mathbf{r}^2}{2m} - \alpha \left(\frac{1}{r} + \frac{b}{p} \right), \quad (3)$$

where α signifies the interaction strength ($Ze^2/4\pi\epsilon_0$ for a hydrogenic atom or GMm for gravity). At this stage we will take the sign of b positive even though it hails from $1/\sqrt{\mathbf{p}^2}$ and we are not entirely sure about the sign of the root because the underlying dynamics (and associated field) is unclear. The choice of sign will become firmer in section 4 but it is fully consonant with the second form of reciprocity substitution. In short, for what it's worth, (3) will be our object of study §. It has the curious, if not dubious, feature that for any finite energy and $r \neq 0$ separation the speed can never vanish; the minimum v may be very small, being determined by the tiny constant b , so it connotes particle restlessness (like zitterbewegung) even at the classical level! The only possibility for the speed to vanish is when the singularity at $r = 0$ is reached. Since r stands for the relative distance between the test body and the centre of influence, it makes more sense to reinterpret $b = m\Omega$ where m is the reduced mass and regard Ω as the truly universal constant. Doing so will ensure that the modified kinetic energy assumes the same form no matter how coordinates are chosen:

$$p_1^2/m_1 + p_2^2/m_2 + m_1\Omega^2r_1^2 + m_2\Omega^2r_2^2 = P^2/M + p^2/m + M\Omega^2R^2 + m\Omega^2r^2.$$

Here R is the centre of mass location, $M = m_1 + m_2$ is the total mass and P is the total momentum. Because of P, R dependence this does not mean that H is translation invariant any more than the usual Coulomb Hamiltonian is invariant under boosts; it is only *covariant* under those transformations. Therefore this represents a philosophical problem for relativity (even Galilean) and we shall worry about it later.

‡ If a Yukawa potential $\exp(-\mu r)/r$ is modified to obey reciprocity, the additional term is $b \exp(-\mu p/b)/p$ and it becomes negligible for small b ; thus nuclear physics is unlikely to be affected.

§ Such a Hamiltonian in turn spawns a strange-looking Lagrangian when it is expressed in terms of position and velocity.

The paper is set out as follows. Section 2 discusses the classical problem and trajectories as b is varied. Not unexpectedly we find that the distorted orbits precess around the force centre and we determine the rate for small b ; this is much like general relativistic corrections to Newtonian gravity. Section 3 deals with the quantum version; we find the change in energy levels to first order in b by the variational method and perturbation theory—which happen to agree with one another. In this way we set limits on the value of b so as not to disturb experimental atomic results, namely $b \leq 10^{-26}$ kg/s, a rather weak conclusion. More stringent limits come by looking at galactic scales, where the $1/v$ term can influence rotation rates profoundly. Section 4 contains our investigations of (3) for galaxies (possessing a supermassive black hole at the centre); there we find that the $1/v$ term can yield a velocity which at first increases linearly from the centre and then steadies out to a constant value, before rising again due to the effect of the harmonic b^2r acceleration. Our conclusions and continued worries with Born’s principle end the paper in Section 5.

2. Classical motion

The equations of motion arising from the non-relativistic expression (3) are

$$\dot{\mathbf{r}} = \mathbf{p}(1/m + b\alpha/p^3), \quad \dot{\mathbf{p}} = -\mathbf{r}(b^2/m + \alpha/r^3), \quad (4)$$

with $H = E$, the conserved “energy”. This means there is a cubic relation between momentum and speed, force and displacement. For positive b the speed/momentum can never vanish, and if $\alpha > 0$ (an attractive interaction) the force cannot disappear either. For the purpose of the ensuing analysis we shall take b to be a small and positive quantity; many of the results can be continued to negative b without endangering the steps and we shall anyway be expressing the (small) precessions of the orbit to first order in b where the sign is somewhat irrelevant.

It is quite difficult to solve these equations in general but it helps to use rotational invariance of (4) and the conserved angular momentum $\ell = |\mathbf{r} \times \mathbf{p}|$ to simplify the Hamiltonian in the usual way:

$$H = \frac{1}{2m} \left[p_r^2 + \frac{\ell^2}{r^2} + b^2 r^2 \right] - \alpha \left[\frac{1}{r} + \frac{b}{\sqrt{p_r^2 + \ell^2/r^2}} \right] = E; \quad p_r \equiv \mathbf{p} \cdot \hat{\mathbf{r}} \quad (5)$$

(Because of reciprocity we could equally well have used $r_p \equiv \mathbf{r} \cdot \hat{\mathbf{p}}$ in place of p_r and p instead of br . However the form (5) is more familiar and this is the framework we shall adopt.) In consequence the radial equations are

$$\dot{r} = \frac{p_r}{m} \left[1 + \frac{\alpha b m}{(p_r^2 + \ell^2/r^2)^{3/2}} \right], \quad \dot{p}_r = \frac{\ell^2}{mr^3} - \frac{b^2 r}{m} - \frac{\alpha}{r^2} + \frac{\alpha b \ell^2}{(r^2 p_r^2 + \ell^2)^{3/2}}. \quad (6)$$

For determining the trajectory in the orbital plane, remember that the rate of change of azimuthal angle is $\dot{\varphi} = \ell/mr^2$, so the trajectory equation is obtained by integrating

$$\frac{d\varphi}{dr} = \frac{\ell}{r^2 p_r [1 + \alpha b m / (\ell^2/r^2 + p_r^2)^{3/2}]}, \quad (7)$$

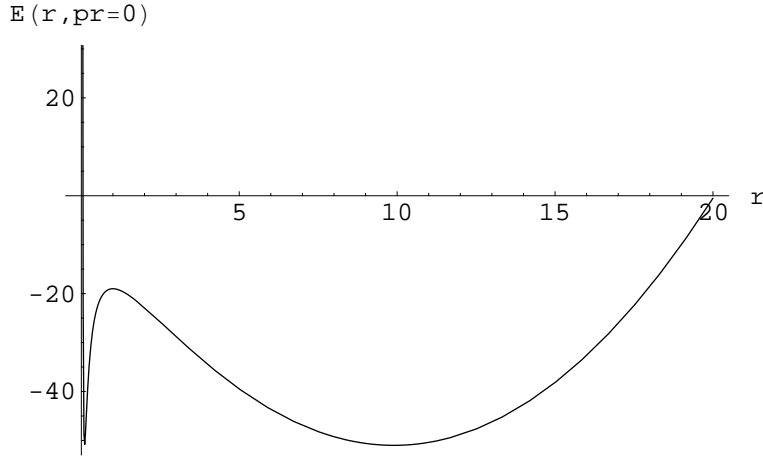


Figure 1. Dependence of energy on radius when the radial velocity vanishes.

in which p_r has to be eliminated in terms of E via (5). This is a hard problem for general b so we shall first turn to Mathematica for some elucidation of the motion and orbits.

By looking at the E -contours in phase space $(r - p_r)$ one finds that for $\ell \neq 0$ the trajectories at lower energy are bounded with $p_r = \dot{r} = 0$ at perigee or apogee. Absolute minima of E arise when $\partial E/\partial r = 0$ or where

$$-\frac{\ell^2}{mr^3} + \frac{b^2 r}{m} + \frac{\alpha}{r^2} - \frac{\alpha b}{\ell} = 0.$$

This cubic equation in r is readily solved and it has three real roots in the region of interest,

$$r_{\pm} = \frac{1}{2b} \left[\frac{\alpha m}{\ell} \pm \sqrt{\frac{\alpha^2 m^2}{\ell^2} - 4bl} \right], \quad r_0 = \sqrt{\frac{\ell}{b}}, \quad (8)$$

with corresponding extremal energy values

$$E_{\pm} = -\frac{1}{2m} \left(\frac{\alpha^2 m^2}{\ell^2} + 2lb \right), \quad E_0 = bl - 2\alpha \sqrt{\frac{b}{\ell}}. \quad (9)$$

E_0 is an unstable maximum, while E_{\pm} are equal value stable minima, and this is best revealed by plotting $E(r, p_r = 0)$ against r in Figure 1. The phase space portrait is drawn in Figure 2 and the corresponding trajectory is depicted in Figure 3 over a timespan of two seconds. In all diagrams we have assumed unit mass and taken exaggerated values, $\ell = 1, \alpha = 10, b = 1$ in order to emphasize the misshapen and precessing orbit.

It is apparent that for $b \neq 0$ orbits become distorted (sometimes very pronouncedly) from Keplerian ellipses and precession occurs. It is of interest to work out the precessional rate for model (3) and small b when the distortions/changes are tiny too.

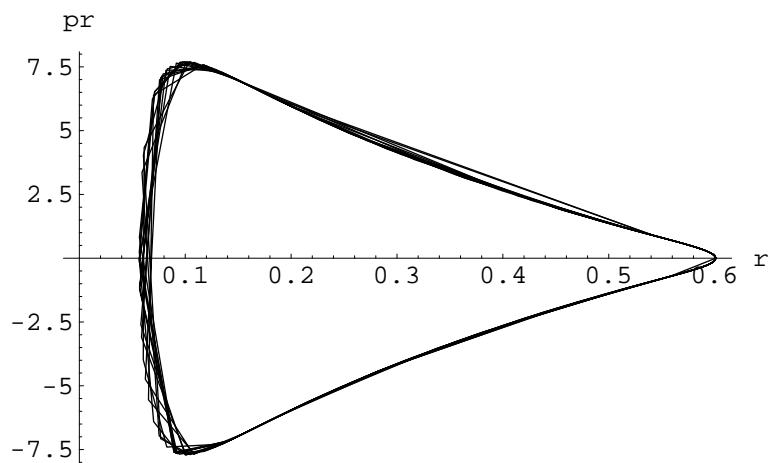


Figure 2. Phase portrait when $E = -21$.

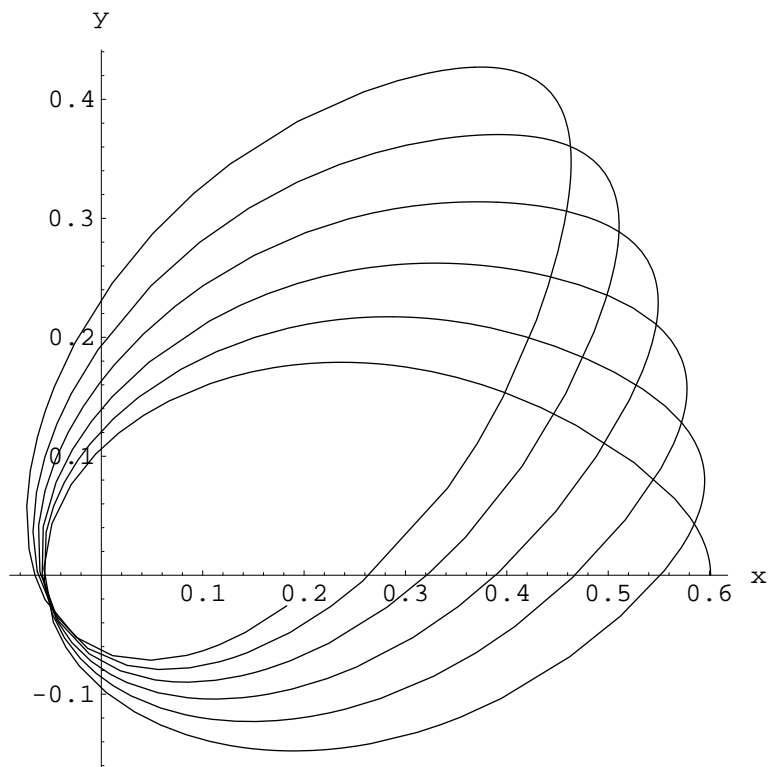


Figure 3. Trajectory when $E = -21$.

To do so we make the standard change of variable $u = 1/r$ and expand (5) and (3) to first order in b . Thus

$$\frac{d\varphi}{du} = \frac{\ell}{p_r} \left[1 + \frac{\alpha b m}{(\ell^2 u^2 + p_r^2)^{3/2}} \right]^{-1} \simeq \frac{\ell}{p_r} \left[1 - \frac{\alpha b m}{(\ell^2 u^2 + p_r^2)^{3/2}} \right], \quad (10)$$

and

$$p^3 - 2m(E + \alpha u)p - 2m\alpha b + b^2 u^{-2} = 0; \quad p = \sqrt{p_r^2 + \ell^2 u^2}.$$

The root we need is $p_r \simeq \sqrt{2m(E + \alpha u) - \ell^2 u^2}$, so expanding around that the trajectory equation (10) simplifies to

$$\begin{aligned} \frac{d\varphi}{du} &\simeq \frac{\ell}{\sqrt{2m(E + \alpha u) - \ell^2 u^2}} \left[1 - \frac{\alpha b m}{[2m(E + \alpha u)]^{3/2}} \right] \\ &= \frac{1}{\sqrt{(u - u_1)(u_2 - u)}} \left[1 - \frac{\alpha b m \ell^3}{[(u_1 + u_2)u - u_1 u_2]^{3/2}} \right], \end{aligned} \quad (11)$$

where $u_1 = 1/r_1, \phi_1 = 0$ at apogee and $u_2 = 1/r_2, \phi_2 = \pi/2$ at perigee – the turning points in r . The first term on the right of (11) produces the usual elliptical orbit answer

$$2u = (u_1 + u_2) + (u_1 - u_2) \cos(2\varphi)$$

and the integration of the second term is connected with the precession. Over an orbit the additional change in azimuth is

$$\begin{aligned} -\Delta\varphi_b &= \alpha b m \ell^3 \int_{u_1}^{u_2} \frac{du}{\sqrt{(u - u_1)(u_2 - u)}} \frac{1}{[(u_1 + u_2)u - u_1 u_2]^{3/2}} \\ &= \alpha b m \ell^3 \int_0^\pi \frac{d\phi}{\frac{1}{2}[(u_1^2 + u_2^2) + (u_1^2 - u_2^2) \cos \phi]^{3/2}} \\ &= \frac{2\alpha b m \ell^3}{u_1^2 u_2} E(1 - u_1^2/u_2^2) = 2\alpha b m \ell^3 r_1^2 r_2 E(1 - r_2^2/r_1^2). \end{aligned} \quad (12)$$

Here $E(k)$ is the complete elliptic integral, which for small argument behaves as $E(k) = (\pi/2)(1 - k/4 - \dots)$ as $k \rightarrow 0$. Hence for small eccentricity ($r_1 \simeq r_2 \simeq a$), the change in azimuth is

$$\Delta\varphi_b \simeq -2\pi\alpha b m \ell^3 a^3, \quad (13)$$

to a good approximation, where a is the semimajor axis. In the following section we shall set limits on b (for electrons at least) such that accurate atomic physics experiments are not substantially disturbed.

3. Quantum mechanical considerations

We will now be dealing with the operator version of (3) for hydrogenic atoms when $\alpha = Ze^2/4\pi\epsilon_0$. and

$$H = \frac{P_r^2 + b^2 R^2 + L^2/R^2}{2m} - \alpha \left[\frac{1}{R} + \frac{b}{\sqrt{P_r^2 + L^2/R^2}} \right]. \quad (14)$$

It is evident that we are dealing with a tricky problem due to the last term, connected with $1/|P|$, even for eigenfunctions of angular momentum $Y_{\ell m}(\theta, \phi)$ so $L^2 \rightarrow \ell(\ell + 1)\hbar^2$ in the Schrödinger equation.

Any serious attempt to try to solve (14) needs an interpretation of P_r^{-1} when $\ell = 0$. As $P_r R(r) \rightarrow (-i\hbar/r)\partial(rR(r))/\partial r$, when acting on the radial part $R(r)$ of the wave function, a reasonable definition of the inverse is the indefinite integral:

$$P_r^{-1}R(r) \rightarrow -\frac{i}{\hbar r} \int_r^\infty dr' r' R(r') \quad (15)$$

for solutions where $rR(r) \rightarrow 0$ as $r \rightarrow \infty$. One readily checks that $P_r P_r^{-1}\psi = P_r^{-1}P_r\psi = \psi$ for normalizable wave functions. It is even true that $P_r^{-1}.\exp(ikr)/r = (1/\hbar k).\exp(ikr)/r$ for outgoing waves, giving extra credence to the interpretation (15).

In the event we have not succeeded in obtaining a complete solution of $u(r) = r\psi(r)$ of type $P(r)\exp(-\kappa r - br^2/2\hbar)$ to the equation for the radial wavefunction

$$-\frac{\hbar^2}{2m} \frac{d^3u}{dr^3} + \frac{d}{dr} \left(\frac{b^2 r^2 u}{2m} - \frac{\alpha u}{r} \right) - \frac{i\alpha b u}{\hbar} = E \frac{du}{dr},$$

because of the troublesome last term on the lefthand side. No matter. One can still obtain a sensible estimate for the change in energy levels to $O(b)$ either by using perturbation theory or making a simple variational calculation. Since the effect is greatest for the ground state, we will analyse the displacement for the lowest energy level both ways – whose answers fortunately agree.

To apply perturbation theory, take the unperturbed wave function in coordinate and momentum space:

$$\psi(r) = \frac{e^{-r/a}}{\sqrt{\pi a^3}}, \quad \phi(p) = \frac{8\sqrt{\pi a^3}}{(p^2 a^2/\hbar^2 + 1)^2}; \quad a = \frac{\hbar^2}{m\alpha} = \text{Bohr radius} \quad (16)$$

and use both to work out the expectation value,

$$\begin{aligned} \Delta E &= \langle b^2 R^2/2m - \alpha b/P \rangle = \frac{2\pi b^2}{m} \int_0^\infty (r\psi(r))^2 dr - \frac{4\pi\alpha b}{\hbar^3} \int_0^\infty p\phi^2(p) dp \\ &= 3b^2 a^2/2m - 16\alpha b a/3\pi\hbar \simeq -16b\hbar/3\pi m, \end{aligned} \quad (17)$$

to first order in b .

With the variational method adopt a trial wavefunction just like eq. (16), except that a is no longer identified with the Bohr radius. Then one gets (as a function of a),

$$E(a) = \frac{\hbar^2}{2ma^2} + \frac{3b^2 a^2}{2m} - \frac{\alpha}{a} - \frac{16\alpha b a}{3\hbar\pi}. \quad (18)$$

Minimising the energy, a cubic equation for the trial radius a is found:

$$\frac{\hbar^2}{ma^3} - \frac{\alpha}{a^2} + \frac{16\alpha b}{3\hbar\pi} = 0, \quad (19)$$

whose solution to order b is $a \simeq \hbar^2/m\alpha + 16b\hbar^5/3m^3\alpha^3$. Hence the optimal energy is obtained as

$$E = -m\alpha^2/2\hbar^2 - 16b\hbar/3\pi m + \dots,$$

thereby agreeing with (17).

Taking the H-atom as the archetypical case we must make sure that atomic values are hardly touched by the reciprocity term ab/P . The Rydberg energy of 13.6... eV $\sim 10^{-18}$ J is known to nine places of decimals and because $\Delta E \sim b\hbar/m_e \sim 10^{-4}b$ J we shall require this to be $\leq 10^{-30}$ J, just to be on the safe side. This sets a limit $b < 10^{-26}$ kg/s. This may seem miniscule but if we translate it into a universal angular frequency via $\Omega \equiv b/m_e$, we obtain a value $\Omega \leq 10^4$ Hz that is not so small. As we shall see presently, galactic considerations produce periods which are *many* orders of magnitude greater; therefore we can state with some confidence that the reciprocity modification with small enough b has essentially no effect on the accuracy of atomic calculations.

4. Reciprocity and rotation of the galaxy

We now treat the gravitational case, based on classical Newtonian dynamics and, presently, its reciprocity variant (3). It is a really important subject as the existence of possible dark matter owes a great deal to this study. In order to make any headway, observations of the motion/distribution of (visible) matter have to be taken into account [3, 4]. It has been established that the Milky Way contains a spinning supermassive black hole at centre with a mass of at least 4×10^7 suns or about 10^{38} kg. Visible mass is about 2×10^{11} suns and on the basis of perceived rotation rates it is inferred that three or four times that mass is hidden in dark matter, as far out as we can see—*assuming ordinary Newtonian* gravity. The rotation speed of the Milky Way [5] seems to increase linearly with radius over a small distance, of about 0.3 kpc $\sim 10^{19}$ m, peaks, oscillates a bit and settles down to some 230 km/s out to 5×10^{20} m from the black hole at the centre. We shall take this as a crude description of our galaxy, neglecting spiral arm structure and the effect of some barring of mass in the middle on the motion. We wish to investigate whether Born's reciprocity modification (3) has anything of consequence to say on the topic of dark matter and, in particular, if the observed tangential velocity profile is consistent with the *visible* matter distribution, without invoking dark matter.

Since we are presuming that the rotation is mainly tangential, if we consider a typical star such as the sun, we may take $p_r \simeq 0$ to a good approximation. So we just need to sum over all other masses M and their relative speeds v to obtain an average value for $\langle \alpha(1/r + b/p) \rangle = \langle GmM(1/r + \Omega/v) \rangle$. Here we have adopted a positive b -sign, with $b \equiv m\Omega > 0$, as before, to fit in with the observations to come; the opposite sign gives nothing but grief.

We cannot perform such averaging without some idea of the mass density distribution $\rho(\mathbf{r})$. *Unmodified* Newtonian gravity indicates that the matter distribution is roughly *spherical* and $\rho(r) \sim 1/r^2$ as far out as one can see, to ensure that the enclosed mass increases linearly with radius and reproduce a constant tangential star speed; hence dark matter. Of course the visible mass distribution flatly contradicts this: it is mostly disk-shaped, with a concentration near the galaxy centre. This is what we

shall model by taking a cylindrical Gaussian distribution,

$$\rho(\varrho, z) \simeq \kappa^2 \delta \mathcal{M} e^{-\kappa^2(\varrho^2 + \delta^2 z^2)} / \varrho \pi^2, \quad (20)$$

to which we shall add a contribution from the central bulge including a black hole. (Here ϱ is the distance from the central axis and z is the distance from the galactic plane.) The integral over the visible mass density produces a finite galactic mass \mathcal{M} ; the parameter κ specifies the size of the galaxy disk while δ is the ratio of disk diameter to disk thickness which varies from about 10 near the centre to 50 at the outer reaches of the disk; so we will take an average value $\delta^2 \simeq 1000$ presently.

Firstly we derive the potential energy in the plane of the disk at location $(\varrho, z = 0)$ for the cylindrical mass approximation above.

$$\begin{aligned} V(\varrho) &= -\frac{Gm\mathcal{M}\kappa^2\delta}{\pi^2} \iiint d\phi dz' d\varrho' \frac{e^{-\kappa^2(\varrho'^2 + \delta^2 z'^2)}}{\sqrt{\varrho^2 + \varrho'^2 - 2\varrho\varrho' \cos\phi + z'^2}} \\ &\simeq -\frac{2Gm\mathcal{M}\kappa^2\delta}{\pi} \iint dz' d\varrho' \frac{e^{-\kappa^2(\varrho'^2 + \delta^2 z'^2)}}{\sqrt{\varrho^2 + \varrho'^2 + z'^2}} \\ &= -\frac{Gm\mathcal{M}\kappa}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u\kappa^2\varrho^2} du}{\sqrt{u(1+u)(1+u/\delta^2)}}, \end{aligned} \quad (21)$$

where we have used the integral representation $2 \int_0^\infty e^{-\xi^2 X} d\xi = \sqrt{\pi/X}$. For large ϱ note that $V(\varrho) \simeq -Gm\mathcal{M}/\varrho$ as expected for a finite-sized source, while for small distances $V(\varrho) \simeq 2Gm\mathcal{M}\kappa^2\delta\varrho$ vanishes as $\varrho \rightarrow 0$. To (21) we will shortly add a contribution from the central bulge, approximated by a mass $c\mathcal{M}$ placed at the middle.

The average over relative velocity v is a lot cruder and relies on data amassed over many years by astronomers. The galactic disk [3] exerts a strong attraction towards the plane on stars which stray from the disk and this results in a velocity dispersion $\Delta v_z \sim 30$ km/s. As we are supposing that the majority of stars are turning at the same orbital speed v , the only relevant quantity is the angle ϕ between the azimuths of the two bodies and the velocity dispersion along the z -axis, $|\mathbf{v}' - \mathbf{v}| = \sqrt{(2v \sin(\phi/2))^2 + 4\Delta v_z^2}$. Therefore the sum simplifies to an integration over azimuth:

$$\int_0^{2\pi} \frac{d\phi}{2\pi|\mathbf{v}' - \mathbf{v}|} \simeq \frac{1}{4\pi v} \int_0^{2\pi} \frac{d\phi}{\sqrt{\sin^2 \phi/2 + \epsilon^2}} = \frac{K(\frac{1}{1+\epsilon^2})}{\pi v \sqrt{1+\epsilon^2}} \equiv \frac{L}{v},$$

where K is the elliptic integral of the first kind and $\epsilon \equiv \Delta v_z/v \sim 0.15$ roughly [3]. (We use L as a parameter to fit the data in due course.) So without much compunction we shall simply take $\langle \Omega/|\mathbf{v} - \mathbf{v}'| \rangle \sim L\Omega/v$ as a fair approximation; after all we are only striving to get an idea of the size of the reciprocity constant Ω here.

Finally there is the matter of the halo contribution. Unlike mainstream ideas inferring dark matter, we rely on observations of *visible* matter (population II stars, white or brown dwarfs, star clusters, etc.) to put a bound of about $c_H \simeq 10\%$ on the amount of halo mass $c_H\mathcal{M}$. Furthermore we shall suppose, like everyone else, that the halo velocity distribution is largely thermal [4]. In order to model these effects we will neglect the small oblateness of the halo and use a radial halo density distribution,

$\rho_H(r) \simeq c_H \kappa_H \mathcal{M} / 2\pi^2 r^2 (1 + \kappa_H^2 r^2)$, multiplied by a normalized velocity distribution $\rho_H(v) = (\beta/\pi)^{3/2} e^{-\beta v^2}$. This fixes the mass of the halo to be $c_H \mathcal{M}$ and the halo velocity dispersion [4] to be $(\Delta v_H)^2 = 3/2\beta$, while the size of the halo is determined by $1/\kappa_H$. It allows us to work out the contribution to the gravitational potential energy due to the halo:

$$\begin{aligned} V_H(r) &= -Gm \left[\frac{1}{r} \int_0^r \rho_H(r') 4\pi r'^2 dr' + \int_r^\infty \rho_H(r') 4\pi r' dr' \right] \\ &= -\frac{Gm c_H \kappa_H \mathcal{M}}{\pi} \left[\frac{2}{\kappa_H r} \arctan(\kappa_H r) + \ln\left(1 + \frac{1}{\kappa_H^2 r^2}\right) \right], \end{aligned} \quad (22)$$

as well as the reciprocity halo contribution:

$$\left\langle \frac{1}{v} \right\rangle_H = \frac{1}{v} \int_0^v \rho_H(v') 4\pi v'^2 dv' + \int_v^\infty \rho_H(v') 4\pi v' dv' = \text{erf}(\sqrt{\beta}v)/v. \quad (23)$$

Putting all this together, and including reciprocity terms, we obtain the total energy dependence on angular momentum $\ell = m\varrho v$ (conserved by axial symmetry) and distance from axis ϱ for a test mass m :

$$\begin{aligned} E(\varrho, \ell) &= \frac{\ell^2}{2m\varrho^2} + \frac{1}{2}m\Omega^2 \varrho^2 - (L+c) \frac{Gm^2 \mathcal{M} \Omega \varrho}{\ell} - Gm\mathcal{M} \left(\frac{c}{\varrho} + \frac{\kappa\delta}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u\kappa^2 \varrho^2} du}{\sqrt{u(1+u)(\delta^2+u)}} \right) \\ &\quad - \frac{Gm\mathcal{M}c_H \kappa_H}{\pi} \left[\frac{2}{\kappa_H r} \arctan(\kappa_H r) + \ln\left(1 + \frac{1}{\kappa_H^2 r^2}\right) \right] - \frac{Gm^2 \mathcal{M} c_H \Omega \varrho}{\ell} \text{erf}\left(\frac{\sqrt{\beta}\ell}{m\varrho}\right). \end{aligned}$$

Since we are presuming that the velocity is largely tangential, $\dot{p}_\varrho = 0 = \partial E / \partial \varrho$, which leads to the force equation:

$$\begin{aligned} 0 &= -\frac{mv^2}{\varrho} + m\Omega^2 \varrho - (L+c)Gm\mathcal{M} \frac{\Omega}{v\varrho} + \\ &\quad + Gm\mathcal{M} \left(\frac{c}{\varrho^2} + \frac{2\kappa^3 \varrho}{\sqrt{\pi}} \int_0^\infty \frac{\sqrt{u}}{1+u)(1+u/\delta^2)} e^{-u\kappa^2 \varrho^2} du \right) \\ &\quad + \frac{2Gm\mathcal{M}c_H}{\pi\varrho^2} \arctan(\kappa_H \varrho) + \frac{Gm\mathcal{M}\Omega c_H}{\varrho} \left[2\sqrt{\frac{\beta}{\pi}} e^{-\beta v^2} - \frac{\text{erf}(\sqrt{\beta}v)}{v} \right]. \end{aligned} \quad (24)$$

We can simplify the look of this equation by rescaling to dimensionless variables. Let $\mathcal{V} = v/\sqrt{G\mathcal{M}\kappa}$, $\mathcal{R} = \kappa\varrho$, $\omega = \Omega/\sqrt{G\mathcal{M}\kappa^3}$, $\mathcal{B} = \beta\kappa G\mathcal{M}$. The tangential velocity profile then reduces to solving an equation for \mathcal{V} as a function of \mathcal{R} :

$$\begin{aligned} \mathcal{V}^2 + \frac{\omega(L+c)}{\mathcal{V}} + c_H \omega \left[\frac{\text{erf}(\sqrt{\mathcal{B}}\mathcal{V})}{\mathcal{V}} - 2\sqrt{\frac{\mathcal{B}}{\pi}} e^{-\mathcal{B}\mathcal{V}^2} \right] \\ = \omega^2 \mathcal{R}^2 + \left[\frac{c}{\mathcal{R}} + \frac{2\mathcal{R}^2}{\sqrt{\pi}} \int_0^\infty \frac{\sqrt{u} e^{-u\mathcal{R}^2} du}{\sqrt{(1+u)(1+u/\delta^2)}} \right] + \frac{2c_H}{\pi\mathcal{R}} \arctan\left(\frac{\kappa_H \mathcal{R}}{\kappa}\right). \end{aligned} \quad (25)$$

The chosen sign for Ω is highly significant in determining the behaviour of the velocity for small \mathcal{R} , viz. $\mathcal{V} \rightarrow (L+c)\mathcal{R}\omega/c$, as required by the data. [Had we reversed the sign of ω or b we would not have been able to fit observations even remotely.]

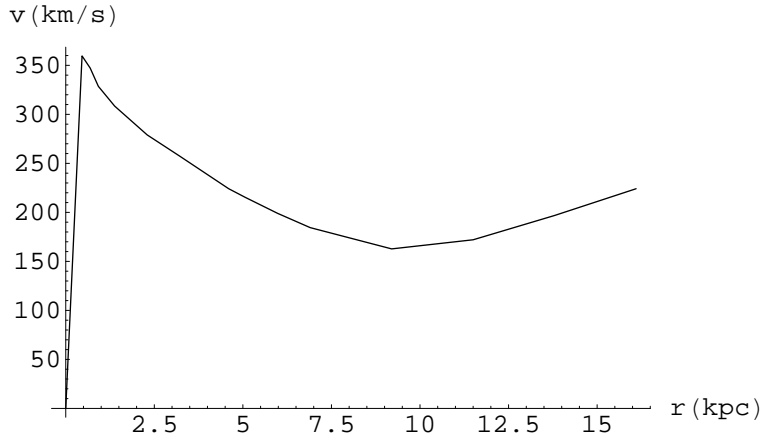


Figure 4. Predicted tangential velocity curve with radius.

A numerical solution of (23) is possible once a few measured values are input. The galaxy is 50000 light-years or more in radius, i.e. about 5×10^{20} m, so let us set the scale $\kappa \sim 0.7 \times 10^{-20} \text{ m}^{-1}$ or so. Taking the visible galactic mass to be roughly $\mathcal{M} \sim 2 \times 10^{41}$ kg, one estimates that $\kappa^3 G \mathcal{M} \sim 5 \times 10^{-30} / \text{s}^2$. Also the rotation speed outside the innermost part of the galaxy equals about 230 km/s and v grows to this value over about 0.3 kpc or 10^{19} m, telling us that $\Omega(L+c)/c \sim 2.5 \times 10^{-14} \text{ s}^{-1}$. In attempting to fit the observed velocity profile, use values $c \simeq 0.05$, as the proportion of galactic mass concentrated in the central bulge, and $c_H = 0.01$ as the ratio of halo mass to disk mass. For simplicity take the halo radius to equal the disk radius or $\kappa = \kappa_H$; although this is an underestimate it makes little difference to the numerical results because c_H is quite small anyhow. A similar comment attaches to our chosen \mathcal{B} -value of about 13 [4]. Finally set $L \simeq 0.85$ and $\Omega \simeq 1.5 \times 10^{-15} \text{ Hz}$, implying $\omega \simeq 0.2$. None of these inputs is at all absurd.

The profile equation (25) now determines \mathcal{V} as a function of \mathcal{R} and the numerical results are plotted in Figure 4. In trying to fit the data it is important to mention that there are two roots for the cubic in \mathcal{V} . In the inner region we use the smaller root, corresponding to the linear relation $v \simeq r\Omega(L+c)/c$, and beyond about $r = 10^{19}$ m we adopt the larger root. Although the Figure 4 graph is rather high below 2 kpc and shows a drop off to steadyish speed that is a bit faster than what is observed, the general shape of the curve is moderately satisfactory. Moreover the flattish part of the curve has $v \sim 200$ km/s, which is the correct magnitude. However it must be admitted that our model and calculation have many rough edges and need a lot more refinement before they can be judged a success; it even conceivable that dressing the model may spoil it rather than enhance it.

5. Conclusions and criticisms

The above fit has some good features and some bad ones; on the positive side it is able in principle to produce an orbital velocity curve that matches the data near the centre (linear rise) and at large distances (almost constant speed), without calling upon a dark matter component — Born’s reciprocity principle has after all modified Newtonian dynamics in a radical way. But it must be confessed that the detailed fit above is far from convincing. This negative feature may be due to the crudeness of our visible mass and velocity distributions. It would be more realistic to include extra matter in the inner galactic bulge, introduce barring plus spiral arms and generally model the velocity distribution more accurately when working out the gravitational potential and its reciprocity counterpart. This is clearly fertile ground for future research. Meanwhile we can probably be content with our *estimate* of $\Omega \simeq 1.5 \times 10^{-15}$ Hz for Born’s hypothesis and not much more; it is truly a galactic scale and corresponds to a period of 1.3×10^8 years, which may be connected with the rotation rate of the galaxy.

Serious concerns remain. The scheme destroys translational invariance because of the way that momentum and position are tied, so a violation of Galilean relativity is to be expected, never mind Lorentzian relativity. Choice of origin is another issue. We have picked the galactic centre as the obvious place; however one might fret about effects of harmonic acceleration ($\Omega^2 \mathbf{r}$) which can get overwhelmingly large from outer regions of the universe, but is somewhat insignificant within a galaxy. Fortunately it appears that on cosmic scales the galaxies are uniformly distributed in every direction all around so one may presume that such attractions will cancel out overall. And because galaxies are receding away from us at Hubble rates the relative velocity term Ω/v diminishes and balances out as one goes out. So it would seem, superficially at least, that one could choose any other galaxy and take the origin at its centre to reproduce its own galactic rotation without worrying unduly about other galaxies.

Fundamental theoretical criticisms can nevertheless be levelled at (3). It is explicitly non-relativistic and should at least be made to conform with special relativity, To carry out that program sensibly one would need to study the quaplectic group [7] and augment the electromagnetic or gravitational field (photon/graviton exchange) by their reciprocity analogues or some other contributions. This signifies overhauling the whole of standard field theory and the task seems rather difficult, if not vague, at this stage. In the end it may turn out that Born reciprocity is unable to fit the data properly or mesh in with our familiar relativistic field theory concepts. Even if it works in limited fashion it would have to be seen as one of a panoply of Modified Newtonian Descriptions (MOND) of gravity over large distances. Taking a skeptical point of view, Born’s idea will probably be found wanting, dark matter will be required and the problem of seeing it by non-gravitational methods will remain with us for a good while. But before the death knell is finally sounded on the subject of reciprocity, specialist galactic modellers need to investigate its ramifications comprehensively.

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References

- [1] Born M 1949 *Rev. Mod. Phys.* **21** 463
- [2] Born M 1949 *Nature* **163** 207
Green H S 1949 *Nature* **163** 208
- [3] Olling R P and Merrifield M R 2001 *Mon. R. Not. Astron. Soc.* **326** 164
Korchagin V I et al 2003 *Ap. J* **126** 2896
- [4] Battaglia G et al 2005 *Mon. Not. R. Astron. Soc.* **370** 1055
- [5] Clemens D P 1985 *Ap. J.* **205** 422
- [6] McClure-Griffiths N M and Dickey J M 2007, submitted to *Ap. J.*
- [7] Low S A 2002 *J. Phys.* **A35** 5711
Jarvis P D and Morgan S O 2006 *Found. Phys. Lett.* **19** 501
Govaerts J, Jarvis P D, Morgan S O and Low S G 2007, *hep-th/0706.3736*