Paper Code: 00667

God-like Educators in a Fallen World

Helen Chick, The University of Melbourne, h.chick@unimelb.edu.au

For many educators, encountering Shulman's 1986 articulation of "pedagogical content knowledge" (PCK) brings an epiphany. It names something we are sure exists, and recognises that "there *really is* something special about what I know for teaching my discipline area". Despite these feelings of resonance and identification, however, debate about PCK in mathematics education and how to develop it in teachers continues. It has proved elusive to define, there have been arguments about how to measure it, and, significantly, showing that it makes a difference to learning outcomes has been difficult, despite the fact that we all feel certain it should and does.

Many of us claim that PCK is central to our teacher education courses. As educators of future mathematics teachers we find ourselves as demigods in a fallen world. The fallen world constrains the design of teacher training programs and the number of contact hours we have; our students' mathematical backgrounds are never quite what we desire; assessment opportunities are limited; and we often have to address content knowledge before we can tackle PCK. The path leading to enlightened mathematics teachers is often thorny and obscure.

Yet even in this fallen world, we demigod-like educators think we have power to direct people on this path to enlightenment. We choose what aspects of PCK to emphasise, based, if you like, on our PCK for PCK. We cannot do all we would like (we are not omnipotent) but our expertise is the basis for choices about what to include. We exhibit a self-asserted omniscience that we are doing what is best.

Yet do we, as demigods, make the same choices? Will future teachers receive the same enlightenment from different demigods? Do different demigods have different expectations about what comprises enlightenment?

PERSONAL BACKGROUND

As a school student I realised that some of my mathematics teachers varied in the extent of their mathematical knowledge, but I was also acutely aware that some were better than others for helping me make sense of mathematics, and that, moreover, it was not always the mathematics experts who were strongest at this. When I became a mathematics teacher myself I became more cognizant of the complexities of teaching and how my mathematical knowledge was necessary but not sufficient for the task of being an effective teacher. It seemed to me that there were extra things that I needed to know, that there was more to knowledge for teaching mathematics than merely knowing mathematics.

When I first encountered Shulman's idea of *Pedagogical Content Knowledge* (PCK)—later in my mathematical and educational career and some years after the oft-quoted 1986 paper—I experienced a distinct feeling of "aha", accompanied with an equally noticeable sense of "well, duh". He had named that ineffable quality that I had known existed but had not been able to pinpoint and articulate. He gave me the language to talk about what it was that I instinctively knew made a difference in the classroom. He allowed me to express more clearly the initially subconscious but key goal that I had decided was critical in my efforts in mathematics teacher education courses: to build teachers' PCK.

There have, however, been several things that have struck me as disconcerting about the nature of PCK, its role in mathematics teaching, and its role in mathematics teacher education. These concerns have existed for a number of years, but they have come to the fore during work with fellow mathematics teacher educators on the Australian Learning and Teaching Council (ALTC) project on *Building the Culture of Evidence-based Practice in Teacher Preparation for Mathematics Teaching* (hereafter referred to as CEMENT). In this paper I want to examine these matters, in order to contribute to our understanding of the work of teacher education. I will begin by discussing the complexity of PCK: for something that appears to resonate so strongly with mathematics teacher educators and researchers, it is still surprisingly elusive.

THEORETICAL BACKGROUND FOR PEDAGOGICAL CONTENT KNOWLEDGE

Following Shulman's definition of *Pedagogical Content Knowledge* (PCK) as "the particular form of content knowledge that embodies the aspects of content most germane to its teachability" (Shulman, 1986, p. 9) there has been no argument about the reality of and need for such knowledge. For teachers it shows that their work involves knowledge specific to the practice of teaching in their discipline area, and for researchers it offers a reason why student performance does not appear to solely on the content knowledge of the teacher (e.g., Begle, 1979; Prestage, 1999).

Yet although most people acknowledged the power of Shulman's term, there have been years of work and argument trying to pin down the specific components that contribute to the knowledge needed for teaching mathematics. Other constructs, such as *Mathematical Knowledge for Teaching* (MKT; see, e.g., Hill, Ball, & Schilling, 2008), have also been proposed. This search for more rigorously defined components is partly driven by the impetus to investigate their impact, which necessitates measuring them. There are challenges here, because PCK (and MKT) are fundamentally concerned with the knowledge used in teaching, and the complexity of classroom milieus makes isolating specific factors difficult. Moreover, Hodgen (2007) found that the knowledge teachers exhibit in classrooms may differ from that demonstrated in tests or interviews. Nevertheless, for the purpose of this paper, three frameworks for investigating teachers' knowledge for teaching mathematics will be discussed.

The work of Ball, Hill, and their colleagues, looked broadly at knowledge for teaching mathematics and distinguished teaching-specific content knowledge—the aforementioned MKT—and PCK, and identified components within these. MKT includes *Specialised Content Knowledge*, mathematical knowledge arising only within the context of teaching; *Common Content Knowledge,* mathematical knowledge used in settings broader than teaching (Ball, Thames, & Phelps, 2008, p. 399); and *Knowledge at the Mathematical Horizon*, which is knowledge of how mathematics topics are related over the span of the curriculum (p. 403). Within PCK there are also three components. *Knowledge of Content and Students (KCS)* is content-related knowledge associated with how students work with the content (Hill et al., 2008, p. 375), such as knowledge of misconceptions and what makes topics difficult to learn (Shulman, 1986, p. 9; Ball et al., 2008, p. 402). *Knowledge of Content and Teaching* incorporates knowledge of how to convey mathematics clearly to others. Content knowledge has crept into these two subcategories of PCK, despite the framework putting content knowledge into MKT separately. Finally, *Knowledge of Curriculum* is knowledge of the resources available for a topic and when to use them.

Hill et al. (2008) examined KCS closely using multiple-choice items in one of the few attempts to measure facets of PCK. Their study revealed that further work is needed to clarify the scope of KCS and to design items capable of measuring it. They found evidence that KCS and subject matter knowledge both contribute to growth in KCS (p. 392-3), which, perhaps not surprisingly, suggests interrelatedness among all the knowledge subcategories.

The PCK framework I developed with colleagues (see Appendix 1 of Chick, 2007; Chick, Baker, Pham, & Cheng, 2006; Chick, Pham, & Baker, 2006) incorporates components of PCK identified in the literature, in more detailed subcategories than the Ball, Hill and colleagues framework. The structure of the Chick et al. framework suggests a continuum for the mix of content and pedagogy. Some components have content and pedagogy tightly linked (e.g., identifying the cognitive demands of a mathematical task); others have content knowledge used within a pedagogical context (e.g., highlighting structural connections among topics); and, finally, pedagogical knowledge can be used in a content context (e.g., using a content-relevant attention-getting activity). Although it lists several categories, the structure is not meant to imply that PCK is discretely compartmentalised, and it certainly suggests that the boundary between content knowledge and PCK is blurred. The components are regarded as "filters" for examining data in order to identify PCK when it occurs.

Finally, the "knowledge quartet" of Rowland, Huckstep and Thwaites (2005) gives a dynamic perspective on subject matter knowledge. The *foundation* component reflects possessed knowledge of mathematics (e.g., knowledge of procedures), with the other components highlighting the *use* of knowledge in the classroom. *Transformation* concerns transforming possessed content knowledge into pedagogically powerful forms, *connection* highlights links among topics and coherence in lesson sequencing, and *contingency* reflects capacity to respond to unanticipated events.

These three perspectives on knowledge for teaching mathematics highlight the complexity of the constructs. Close examination of the definitions suggests links and overlap among components of the different frameworks, but also reveal that one framework may identify—or at least emphasise—characteristics absent from—or underemphasised in—another scheme. Although researchers have used these different ways of compartmentalising knowledge for teaching mathematics it is not clear whether a compartmentalised view of this knowledge is an accurate reflection of how it plays out in practice. Watson (2008) is one, at least, who has argued that compartmentalisation distracts us from the relationships among components. Nevertheless, although the abundance of frameworks and the overlaps among them can be frustrating at times, they at least reassure us that PCK is a definite construct, even if it might be a bit fuzzy around the edges.

PCK IN MATHEMATICS TEACHING

Our interest in PCK is grounded in the assumption that PCK *is* important for mathematics teaching. We want teachers to have high levels of it, which means (a) we are interested in measuring it, and (b) we are interested in developing it, particularly for pre-service teachers (our focus in the CEMENT project and here), but also through professional learning in-service programs.

As suggested above, measuring PCK has proved difficult. The multiple choice items used by Hill, Ball, and colleagues, for examining knowledge of content and students (a component of PCK that other frameworks suggest might be multifaceted) yielded some results and have the advantage of being able to be administered to large groups at small cost. However, they do not give detailed insights into the reasons behind the pedagogical choices prompted by the items, and do not discriminate among different "levels" of PCK as well as the researchers had hoped.

My own work in this area (e.g., Chick, Baker, et al., 2006; Chick, Pham, et al., 2006) did not attempt to measure PCK, but to examine it through the filters of the different categories of knowledge that seemed significant. The mechanism for gathering data was a questionnaire and follow-up interview, which allowed teachers to explain and justify their pedagogical decisions for situations posed in the questionnaire. While this approach allowed a more detailed examination of different facets of PCK, it is not cost-effective for larger groups, since each interview was up to an hour's duration.

Despite the fact that measuring PCK is proving difficult, decades of mathematics education research has revealed considerable information about facets of PCK, such as students' understanding, and the effectiveness of different material models for concepts. We thus know a great deal about what contributes to PCK in terms of, for example, knowledge of students' thinking and knowledge of representations. As an illustration that is particularly relevant for our later discussion, Stacey, Helme, Condon, and Archer (2001), highlighted the idea of "epistemic fidelity" for models, which is the extent to which the materials and actions on a model match the mathematical concepts they represent.

Based on this knowledge, we *think* we know what aspects of PCK are important in mathematics education, and what ought to be emphasised in courses that prepare mathematics teachers. As teacher educators most of us make decisions about what to include in our courses based on our own acquired PCK and knowledge of research. We do so in the belief that this knowledge is important for our future teachers.

PCK IN MATHEMATICS TEACHER EDUCATION

What is interesting is that we as teacher educators do not often have chances to discuss our choices. We make many of our choices in isolation, perhaps with some advice from colleagues if we are lucky enough to have other mathematics educators in our institutions. We may inherit course descriptions (if, indeed, we are not the ones actually writing them), but these are usually vague enough that we still end up with plenty of leeway to decide what to emphasise and how, *and the decisions are ours*.

We end up becoming god-like arbiters of what will be included in mathematics education courses, with the power—we would like to think—to make a difference to the quality of teachers produced by them. I do not believe that any of us claim to be omniscient (so I will refer to us as "demigods", and I will return to this later), yet there *is* an element of omniscience exhibited by the fact that *we* are making choices about what aspects of the PCK gospel to preach to our neophytes. However, the very necessity of these decisions implies that we are working in a fallen world. If we were in the mathematics teacher education equivalent of the garden of Eden, we would be able to cover all aspects of PCK while working with fully-formed mathematics experts, though not yet expert teachers. Instead our neophytes come to us with varied backgrounds, and we have limited contact hours in which to convey the glorious gospel of PCK.

In fact, in this article I have positioned my pre-service teachers as students, and myself as their demigod teacher, and I have done this because I do view my pre-service teachers as learners. I am not denying that they are also teachers in embryo, and that my role is to help them through this gestation period, but here the demigod metaphor implies and reflects the fact that there *is* knowledge possessed by an expert that is to be imparted, and "students" reflects their role as potential recipients of what we offer. They are by no means "disciples", although some of them become converted, while perhaps a few remain if not atheistic then at least agnostic!

So, given a pantheon of mathematics educators, presumably with similar knowledge about what constitutes useful PCK, is it true or likely that they will make the same decisions about what to include in their courses? Already this seems implausible, because (with apologies to Tolstoy) while gardens of Eden are all alike, every corner of the fallen world has fallen in its own way. This became very apparent to us in the early days of the CEMENT group, as we discussed the diversity of constraints upon our teacher education programs: no two universities had much in common at all. Numbers of subjects, modes of offering (e.g., face-to-face or online), contact hours within subjects, academic backgrounds of students, duration of the whole course, and so on, all varied wildly. We were not surprised, then, to discover that we had had to make different choices about what aspects of PCK to emphasise.

As the work of the CEMENT group continued, with its intention of trying to explore what approaches might make a difference to building PCK in pre-service teachers, we needed to start designing items that would allow PCK to be investigated, reservations about this notwithstanding. We had wide-ranging discussion about what the items were trying to capture, and how best to get at PCK through on-line items that were to be scored automatically, when PCK is fundamentally about actual teaching decisions in the real and complex milieu of learners known and unknown.

As we considered the breadth and depth of the items, we were struck by three things. First, even with such an elusive construct we all seemed able to "recognise PCK when we saw it" and concur about when it was and was not present in items. Second, it is hard to design PCK items. We were not surprised by this (Hill et al, 2008 give an interesting account of their difficulties), but the process of trying seemed to improve our own understanding of PCK (I will discuss this further in the conclusion). Finally, it became apparent that some of us, at least, had certain "pet" aspects of PCK that are particularly emphasised in our courses, thus confirming our suspicions that demigods do act differently. This is highlighted by one incident in our discussion that I would like to examine in more detail.

THE DISCRETE FRACTIONS SCENARIO

During the course of our discussions about items for our online PCK instruments, there were many items that provoked intense debate about the content, focus, wording, and structure of the item. There was one particular item, however, that epitomised the complexity of the issues raised here. It was intended for primary pre-service teachers, and depicted two fractions in a visual representation, showing each of the two fractions as proportions of a discrete set. Pre-service teachers were then asked which of four diagrams would best show the child how to see the sum of the two fractions.

I can only write in detail about the reasons for my own reactions to this item, but it is safe to say that there were passionate reactions from my colleagues on the project, and I *can* report on some of their comments. For myself, I did not like the item at all. I had two reasons for this. The first was recognition, based on my own pedagogical content knowledge of the suitability of different models for fractions, that discrete representations are problematic, particularly for operations with fractions. When I raised this objection, my colleagues agreed but then countered with the argument that some students will self-select a discrete representation anyway, and teachers will have to deal with this. My argument against the item was based on my "knowledge of representations", as a sub-component of PCK; my colleagues' arguments *for* the item were based on the idea that it examined "contingency", as part of Rowland et al.'s (2005) knowledge quartet.

There was a part of me that could accept the validity of my colleagues' contention, but there was another part of me that still felt uncomfortable. This time it was because I was worried about how "my" pre-service teachers would do on such a question (good demigods worry about their charges!). Since I know about the problematic nature of discrete representations, I mention them only briefly in my primary mathematics education subject and spend most of my fractions time on representations with greater epistemic fidelity (Stacey, et al., 2001), such as linear and area models. As demigod in my own little corner of the fallen world, I have had to choose what to emphasise, and with only 18 hours to cover the entire primary number curriculum, discrete representations get very short shrift, and "contingency with discrete representations" does not even get a look in.

The problem, of course, concerns the fact that "contingency" is something that is difficult to teach in any case. As teacher educators, we may take time in our courses to address a few typical incidents that may arise in classrooms in certain contexts and with certain topics—indeed, knowledge of typical conceptions is a widely recognised part of PCK—but we know that children have the capacity to come up with the unexpected. In this case, addressing the contingency of this particular awkward representation requires of the teacher a deep and explicit understanding of the role of the whole, a concept that is more implicit and natural in other models. So, it is conceivable that, as a demigod leading my students to enlightenment and having emphasised the linear and area models for fractions, I have given my students capacity to use good models effectively but have diminished their capacity to deal with this specific contingency of a discrete model. To complicate matters, the particular questionnaire item being proposed was a multiple choice item, with some distractors involving linear and area models, and I could envision that the positive attention I had given these in class might distract my students from the “correct” response.

Moreover, I can imagine what might happen should one of my students face such a situation in practice, not least because I have seen myself act similarly to the way I am about to describe. If I have had any effect at all as a demigod, then my student might respond in the following way to the child using the discrete representation of fractions for addition: “Hmmm, this fraction addition would be so much easier to talk about if we used a different model (yes, one of those really excellent models that Helen drummed into us in class!)”. The student might then proceed to encourage the child to abandon the discrete model for a “better” linear model, and proceed to give a clear, appropriate, and enlightening explanation using the materials and language that had been discussed in our pre-service course. The child may well depart with a good understanding of fraction addition, and an observer may even praise the efforts of my student and acknowledge the PCK that has been demonstrated.

In so doing, however, my student has *not* demonstrated another aspect of PCK: knowledge of children’s thinking. My hypothetical student has not been able to work from the child’s current understanding, and has denied the validity of the child’s discrete model (and it *is* valid, just less satisfactory for certain fraction concepts). The student’s reaction to the contingency of a child with a difficult representation was to change the representation rather than to work with the child’s. A different outside observer may well take a view opposing the first outside observer, and claim that my student has shown low levels of PCK because of the failure to understand and work with the child’s current understanding.

If we are going to claim that PCK is a valuable thing, and attempt to measure it, then we have to reconcile these two opposing viewpoints. The problem is that we can only judge an individual’s PCK based on inferences made from what we have observed. Even in the case of my hypothetical student working with the child in some imagined real practice (!), it is still not easy to identify the reason for the action by the action alone, and often it is the reason that tells us most about PCK. My hypothetical student could choose to use a linear model wutht the child instead of a discrete model for any of the following reasons:

(i) “Helen said that linear models were best for fractions, and since Helen is a demigod I will obey her good word; therefore, dear child, I will impose linear models upon you”

(ii) “Oooh, dear child, I have no idea how to come up with the right answer for addition with that discrete model since we didn’t cover that in Helen’s class (it was *such* a rushed course); so I think it would be good if I introduced you to linear models, because I know I can explain them”

(iii) “Hmmm. I think I can explain how to do addition with this model, dear child, but it is going to require such a careful conceptualisation of what is the whole and what the parts are, and what it means to add them together, which I *could* do based on Helen’s stuff about how fraction understanding requires clear identification of the whole, but it’s going to be very demanding for you, and it’s much easier to conceptualise what’s going on if I use a linear model”

(I have to say that I would be watching with bated breath should my student actually decide to acknowledge the child’s discrete model and attempt to work with it to explain fraction addition. It is something I would be very cautious about attempting myself, because it *does* require “such a careful conceptualisation of what is the whole and what the parts are, and what it means to add them together". And all of this is overlooking the very real possibility that my student might work with the discrete model, and get it totally wrong.)

At the end of the day, in cases (i) to (iii) the observed action is the same—teach using the linear model—but I would contend that there are different levels of PCK evident in the responses. Unless we ask students/teachers the basis for their decision we do not always get the full picture of their PCK. As for what we can conclude on the basis of a response to the multiple choice item, when it is difficult to observe very much at all apart from a simple choice, we have to proceed with caution. Indeed, this has been one of the main concerns raised about past items for examining PCK, and was at the fore of the issues considered by the CEMENT workers as the team tried to design our own items. This argument is beginning to suggest that we cannot deduce much about PCK from observation alone, but this is not my contention. I am merely highlighting that there can be circumstances in which the results are not clear cut, and why the design of items is so very difficult.

In fact, the example above highlights another aspect of the fallen world. We demigod educators deliver our gospel to our students, and, verily, some days the message even comes from our lips in pure and powerful form (I will concede that some days are better than others; we are only *demi*gods after all!). My hypothetical student above was, it seems, at least paying attention to and making sense of my pronouncements from on high; hence, the decision to use the linear model. What is sometimes striking, however, is the students' capacity to hear our message in different ways, to think we said one thing when we did not, or to interpret it to mean something other than what we intended. I have had students "cite" me in their assignments, and wondered if we each had the same set of stone tablets as our reference. Our students—inadvertently I am sure—interpret the gospel in non-canonical ways, in some cases hearing but not understanding, even when we are not talking in parables. I can imagine students taking my assertion that some fraction models are better than others and distorting it into "Helen said that the other models are wrong and should never be used". The "thou shalt nots" are often so much easier to understand; yet the complexity of learning and teaching requires PCK that allows deep consideration of principles that appear to be competing (as in (iii) above). And to complete the mix, mathematics itself adds extra complexity with its rich interconnections and the tensions between the development of procedural and conceptual understanding.

CONCLUDING COMMENTS

Is all of PCK teachable? In a fallen world, the answer is clearly no. The demigods are forced to prioritise because of the fallen world's constraints, and our mortal students sometimes corrupt our messages. What most of us do, I suspect, is try to teach correct principles and hope that students can generalise and develop their own knowledge based on those principles in cases of contingency. What sometimes happens, however, is akin to what sometimes happens in children's mathematics classes: either the teacher or the learners themselves want to reduce the domain to a set of "factoids"—a collection of disconnected cases for dealing with any given situation. We run the risk of ending up with a litany of rules like "a negative times a negative is a positive" or "when comparing ragged decimals equalise the lengths with zeros", or, in the domain of PCK, "when adding fractions that are modelled as discrete sets you must …". Just as in the former cases (teaching mathematics) this leads to impoverished mathematics learning, it seems likely that such an approach in teacher education will lead to impoverished development of PCK. The alternative, however, may mean that students might encounter certain special cases—such as adding fractions represented discretely—and struggle to deal with them. All I can hope, as the demigod responsible for the choice of approach, is that they have sufficient principles to be able to make a sensible attempt, built on appropriate pedagogical and mathematical knowledge.

I explained earlier why I positioned my pre-service teachers as students: they are neophytes and acolytes on the journey to teacher-hood, who, I hope, give some attention to me as their demigod source of wisdom (I don't think they actually worship me!). In fact, I view myself as a learner too, despite my demigod status. My doctorate is in mathematics and I taught tertiary calculus and algebra, yet I know how much I have learned about PCK for primary mathematics teaching over the past 12 years. In addition, I have learned a great deal about how to teach this PCK to primary mathematics teachers—PCK for teaching PCK, if you like. More recently I have become more actively involved in secondary mathematics again (the area in which I started my teaching career), and I am sensitive about how much I have to learn about PCK for this area, because I now realise that I knew so little as a beginning teacher. Even more worrying, though, is how much I feel I have to learn about how to convey this PCK to my own pre-service secondary teachers.

This issue—of developing our own PCK-for-teaching-teachers so that we know how best to help our pre-service teachers learn PCK-for-teaching-children—has been one of the features of our discussions in the CEMENT pantheon. It is rare to experience a professional development program for mathematics educators, and the ALTC project has allowed us to do a little of this for ourselves. Following our work on designing the PCK items we all expressed the sentiment that we had experienced a sense of exhilaration from the process of having conversations about our work as mathematics teacher educators. The demigods are still learning, and we relished the opportunity for professional growth associated with building PCK-for-teaching-teachers.

REFERENCES

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*, 389-407.

Begle, E. G. (1979). *Critical variables in mathematics education: Findings from a survey of the empirical literature*. Washington, DC: Mathematical Association of America and National Council of Teachers of Mathematics.

Chick, H. L. (2007). Teaching and learning by example. In J. Watson & K. Beswick (Eds.), *Mathematics: Essential research, essential practice* (Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia, pp. 3-21). Sydney: MERGA. From: http://www.merga.net.au/publications/counter.php?pub=pub_conf&id=369

Chick, H. L., Baker, M., Pham, T., & Cheng, H. (2006). Aspects of teachers' pedagogical content knowledge for decimals. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 297-304). Prague: PME.

Chick, H. L., Pham, T., & Baker, M. (2006). Probing teachers' pedagogical content knowledge: Lessons from the case of the subtraction algorithm. In P. Grootenboer, R. Zevenbergen, & M. Chinnappan (Eds.), *Identities, cultures and learning spaces* (Proceedings of the 29th annual conference of the Mathematics Education Research Group of Australasia, pp. 139-146). Sydney: MERGA.

Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education, 39*, 372-400.

Hodgen, J. (2007). The situated nature of mathematics teacher knowledge. Paper presented at Nuffield seminar series on *Mathematics Knowledge in Teaching* (Seminar 1, 11-12 January 2007). Accessed 17 Nov 2009 from http://www.maths-ed.org.uk/mkit/Hodgen%20Nuffield%20Jan%202007.pdf

Prestage, S. (1999). *An exploration of teachers' mathematical subject knowledge.* Unpublished doctoral thesis. London: King's College, University of London.

Rowland, T., Huckstep, P., & Thwaites, A. (2005). Elementary teachers' mathematics subject knowledge: The knowledge quartet and the case of Naomi. *Journal of Mathematics Teacher Education, 8,* 255-281.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15* (2)*,* 4-14.

Stacey, K., Helme, S., Archer, S., & Condon, C. (2001). The effect of epistemic fidelity and accessibility on teaching with physical materials: A comparison of two models for teaching decimal numeration. *Educational Studies in Mathematics, 47*(2), 199–221.

Watson, A. (2008). Developing and deepening mathematical knowledge in teaching: Being and knowing. Paper presented at Nuffield seminar series on *Mathematics Knowledge in Teaching* (Seminar 5, 18 March 2008). Accessed 10 Oct 2009 from http://www.maths-ed.org.uk/mkit/MKiT5_Watson_distribution-version.pdf .